



OPTION PRICING WITH TIME-CHANGED FRACTIONAL BROWNIAN MOTION: A FRACTIONAL VARIANCE GAMMA MODEL

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ABSTRACT. Fractional Brownian motion (fBm) exhibits attractive features for financial modeling, including long-range dependence, path roughness, and anomalous diffusion. However, its non-semimartingale nature precludes the use of conventional no-arbitrage approaches to option pricing. We address this limitation by introducing a time-changed fBm, obtained by evaluating fBm at stochastic gamma activity time, where activity time represents cumulative executed trading time. The resulting process retains the defining properties of fBm while recovering the semimartingale structure. Building on this construction, we develop the fractional Variance Gamma (fVG) model and propose a generalized method of moments (GMM) estimation procedure for option pricing. An empirical analysis of the S&P 500 yields an estimated Hurst exponent of approximately 0.47, consistent with mildly sublinear temporal scaling of return moments.

1. Introduction. While Brownian motion remains the canonical continuous-time stochastic process for modeling financial time series, it does not capture several empirical regularities documented in financial markets. Fractional Brownian motion (fBm) generalizes classical Brownian motion by accommodating long-range dependence, path roughness, and anomalous diffusion, allowing fBm-driven processes to incorporate empirical features that are absent from the classical Brownian setting.

One important application is option pricing, where stock prices serve as the underlying asset. Several empirical studies report evidence of long-range dependence in stock returns, challenging the short-memory assumption embedded in standard diffusion models [65, 22, 9]. In addition, a growing body of evidence documents multifractal scaling in asset returns across temporal horizons, e.g., [43, 4, 5, 14, 58, 62]. For instance, [46] show that the distribution of S&P 500 returns measured over different time intervals satisfies a power-law scaling relation with a Lévy exponent of approximately 1.4. Complementing this evidence, [38] reject the random walk hypothesis using the variance-ratio tests, finding that the variance of multiperiod returns grows more than linearly with the sampling horizon. Similar findings have

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subsequently been reported for Asian [6, 27, 53], European [56], and African equity markets [55]; see also [39, 6].

However, fBm is not a (semi)martingale, which presents a fundamental obstacle when applying the standard continuous-time, continuous-trading, arbitrage-free pricing framework.¹ Specifically, the fundamental theorems of asset pricing rely critically on the semimartingale property of the underlying asset. When the price process is not a semimartingale, Itô calculus no longer applies [50], and arbitrage opportunities inevitably arise in the standard frictionless setting [13]. Consistent with this result, [52] and [8] show that the fractional Black–Scholes–Merton model admits arbitrage under continuous trading. Section 2 provides a comprehensive review of fractional stock price models.

In this paper, we propose a novel approach to option pricing by introducing a time-changed fBm process, defined as $X = B_H(\gamma)$, where the fBm B_H is evaluated at stochastic gamma activity time γ . The use of activity time has a long tradition in option pricing; see, e.g., [40, 21, 42]. Economically, activity time represents the accumulation of trading activity and the arrival of market information. Modeling asset dynamics in activity time rather than calendar time therefore provides a natural way to account for variations in trading intensity. In this view, price movements are driven by the arrival of new information through trading activity rather than solely by the passage of calendar time.

The time-changed fBm preserves the principal statistical properties of standard fBm while remaining analytically tractable. In particular, its distributional properties can be characterized through a Gaussian–Gamma mixture representation. More importantly, our central theoretical result establishes that the time-changed fBm is a semimartingale, rendering it compatible with the classical no-arbitrage pricing framework based on equivalent martingale measures. Although a gamma activity process is adopted in the present exposition, our main results extend more broadly to any pure jump, finite-variation activity-time process with α -bounded variation on compact sets for every $\alpha > 0$.

Building on this result, we introduce the fractional Variance Gamma (fVG) model, which extends the Variance Gamma (VG) model of [41] by replacing the Brownian motion component with fBm. The resulting five-parameter specification captures the drift, volatility, skewness, kurtosis, and dependence structure of asset returns. Unlike the classical VG model, however, the fVG process does not possess independent increments and does not admit a closed-form density. Consequently, standard pricing and estimation methods that rely on independent increments or closed-form expressions are no longer directly applicable. To address these challenges, we develop a marked point process representation of the time-changed fBm, characterize its compensator, and derive closed-form expressions for its raw and central moments.

Under the fVG model, we show that stock returns admit a decomposition into four distinct components: (i) compensation for the risk-free rate, (ii) a risk-premium, (iii) an endogenous drift generated by the dependence structure and historical trajectory of the process, and (iv) a martingale innovation that captures residual uncertainty. Unlike classical semimartingale models, the fVG stock price specification therefore contains a dependence-induced drift through which the current state of the process influences its future evolution. This provides

¹See [32, Chapter 2] for a textbook presentation of the standard arbitrage-free pricing model.

an economically interpretable channel through which persistence in returns affects asset prices. We further consider an alternative stock price specification based on time-changed fractional geometric Brownian motion and establish its equivalence to the proposed fVG model.

To compute the model’s conditional expectations, we construct a simulation procedure that first generates the discretized sample paths of the fBm and gamma processes independently, and then evaluates the fBm at the realized gamma activity times. To ensure numerical accuracy, the fBm is generated on a finer (secondary) grid than that used for the gamma activity time (primary grid).

For parameter estimation, we propose a generalized method of moments (GMM) procedure for the fVG model that exploits the unconditional moments of stock returns across multiple time horizons. To illustrate the proposed procedure, we fit the fVG model to the S&P 500 index and compare the full specification with several nested benchmark models, including the classical Black–Scholes–Merton (BSM), Variance Gamma (VG), and fractional Black–Scholes–Merton (fBSM) models. Strikingly, the estimated Hurst parameter H under the full fVG specification is approximately 0.47, which lies below the Brownian benchmark of 0.5.

This result contrasts with studies reporting long-range dependence in asset returns. It suggests that some of the apparent long memory documented in simpler models may partially reflect unmodeled heavy-tailed behavior rather than genuine temporal dependence. Our GMM estimator identifies H primarily through the scaling behavior of return moments across investment horizons, that is, through the rate at which these moments accumulate with lag length. Consequently, the estimate $H \approx 0.47$ implies that the term structure of empirical moments exhibits mildly sublinear temporal scaling.

Overall, the proposed fVG model accommodates fractional dynamics in stock price evolution while preserving compatibility with the classical arbitrage-free pricing paradigm. More broadly, the underlying time-changed fBm provides a flexible stochastic process for a broad range of financial time series applications. Although time-changed fBm has appeared in the statistical literature, to the best of our knowledge, this is the first paper to incorporate such a process into an arbitrage-free asset-pricing framework. We leave a comprehensive empirical evaluation of the fVG option pricing model to future research.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 provides background on the stochastic processes underlying our analysis and establishes key results used throughout the subsequent analysis. Section 4 introduces the time-changed fBm. Section 5 develops the fVG equity model. Section 6 presents a GMM procedure for parameter estimation. Section 7 reports an empirical application to the S&P 500 index. Section 8 concludes. Proofs of the main theorems and propositions are collected in Appendix A.

2. Literature review on fractional asset pricing models. Modern continuous-time, continuous-trading option pricing theory is built on the assumption that asset prices follow semimartingale processes. This assumption is fundamental because semimartingales constitute the largest class of stochastic processes for which Itô calculus is well defined (e.g., see [50]), providing the mathematical foundation for much of continuous-time financial economics. Although fBm possesses several empirically appealing properties — including long-range dependence, path roughness,

and anomalous diffusion — it is not a semimartingale. Over the years, several approaches have been developed to incorporate fBm into financial modeling.

The earliest attempts adopted a direct substitution approach, replacing Brownian motion with fBm in the classical Black–Scholes–Merton (BSM) framework to obtain a fractional BSM model. However, this formulation was shown to be incompatible with the no-arbitrage framework of continuous-time finance. Under the standard continuous-time, continuous-trading setting, the fundamental results of [13] establish that arbitrage-free pricing requires the underlying price process to be a semimartingale. Consistent with this result, several studies have constructed explicit arbitrage strategies for fBm-driven asset prices, demonstrating that the fractional BSM model admits arbitrage under continuous trading [52, 8].

To circumvent the arbitrage issues arising from the non-semimartingale nature of fBm, a second approach modifies the definitions of arbitrage and self-financing trading strategies to develop an analytically tractable pricing framework based on divergence/Wick calculus [28, 15]. However, these modified concepts fail to preserve the standard economic interpretation of portfolio values and trading strategies, limiting their applicability and relevance in financial valuation [3].

Separately, an alternative approach in the literature incorporates fBm into asset pricing models by modifying the frictionless market assumption. The central idea is to restrict the class of admissible self-financing trading strategies to restore the absence of free-lunch-with-vanishing-risk (FLVR). For example, [8] and [33] consider trading strategies that impose a strictly positive minimum time interval between consecutive trades. [2] restrict admissible strategies to those representable as Borel-measurable functions of a finite, prespecified collection of adapted functionals of the stock price path. Alternatively, [24] introduces proportional transaction costs and restricts trading strategies to processes of finite variation. While these approaches successfully restore the FLVR condition, they do so by departing from the classical frictionless continuous-trading framework. Consequently, the Fundamental Theorems of Asset Pricing no longer apply directly, and option valuation cannot generally be formulated through equivalent martingale measures. This departure limits the analytical tractability of these approaches.

Another branch of the literature constructs alternative semimartingale processes that replicate selected features of fBm, such as long-range dependence and self-similarity. Examples include time-changed Brownian motion with long-range dependent activity times [4, 26, 16, 17, 18] and regularizations of fBm [7, 52]. While these approaches retain compatibility with semimartingale-based asset pricing, they typically introduce additional modeling complexity and depart from the canonical Brownian-motion-based paradigm.

Finally, [61] develop an arbitrage-free valuation framework for derivatives written on non-traded fractional processes. Their approach establishes a coherent pricing and replication theory for derivatives written on a non-semimartingale underlying, provided that the underlying process itself is excluded from direct trading. Arbitrage-free valuation is achieved through trading in a sufficiently rich set of derivatives written on the underlying, allowing the standard arbitrage-free pricing methodology to be applied despite the non-semimartingale nature of the underlying process. However, this line of work does not permit the prices of traded assets themselves to exhibit fractional dynamics.

In contrast to the existing literature, our stock price process generated by time-changed fBm preserves the principal statistical properties of standard fBm, including its ability to generate long-range dependence, irregular sample paths, and anomalous diffusion across parameter regimes. Moreover, its Gaussian–Gamma mixture representation yields a tractable characterization of the process that facilitates both theoretical analysis and statistical implementation. At the same time, it remains fully compatible with the classical semimartingale framework for continuous-time arbitrage-free asset pricing through the use of stochastic activity time, a well-established concept in the option pricing literature.

3. Stochastic processes. This section introduces the three stochastic processes underlying the fractional Variance Gamma (fVG) model: market point processes, the gamma process, and fractional Brownian motion (fBm). It also establishes several technical results that are used throughout the subsequent development of the model. Throughout the paper, we work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual hypotheses, where $\mathbb{P}$ denotes the statistical probability measure and $\mathbb{F}$ is the joint (enlarged) filtration generated by the constituent processes, the gamma activity time γ and fBm process B_H , introduced below. That is, $\mathcal{F}_t = \mathcal{F}_t^\gamma \vee \mathcal{F}_{\gamma_t}^{B_H}$. For notational convenience, we suppress explicit dependence on $\omega \in \Omega$ and model parameters whenever no ambiguity arises.

3.1. Marked point process. Since all pure jump processes can be represented as marked point processes, we provide a brief heuristic overview of these processes for readers less familiar with them. For a more comprehensive treatment, we refer the readers to standard textbooks, see, e.g., [11, 10].

Pure jump processes are fully characterized by their jump structure. In particular, each jump is identified by its occurrence time and corresponding jump size. While for finite-activity jump processes it is feasible to represent the dynamics explicitly through a sequence of jump-time and jump-size pairs, $\{(T_1, K_1), (T_2, K_2), \dots\}$, this representation becomes inadequate in the case of infinite-activity processes, where infinitely many jumps may occur over any compact time interval.

To accommodate this general setting, a more flexible measure-theoretic formulation is required, in which jump times and magnitudes are jointly encoded through random measures that track the full jump structure of the process. Consider a finite-activity marked point process Y with associated jump measure

$$\mu_Y(dt, dx) = \sum_{s: \Delta Y(s) \neq 0} \delta_{(s, \Delta Y(s))}(dt, dx),$$

where δ denotes the Dirac measure on $\mathbb{R}_+ \times \mathbb{R}$ and $\Delta Y(s)$ represents the jump size (mark) of the process Y at time s . Intuitively, the random measure μ_Y encodes the jump times and jump size pairs of the process Y . Conversely, given the jump measure μ_Y , the finite-variation process Y can be recovered via integration against the identity mark function,

$$Y(t) = \int_0^t \int_{\mathbb{R}} x \mu_Y(du, dx).$$

However, the jump measure μ_Y only encodes the realized jump occurrences and does not directly capture the probabilistic structure of the process. To characterize its distributional properties, it is more convenient to work with its compensator. ν_Y

is defined to be the compensator (measure) of μ_Y if there exists a unique predictable random measure ν_Y such that, for every Borel set $B \subseteq \mathbb{R}$, the process

$$t \mapsto \mu_Y((0, t] \times B) - \nu_Y((0, t] \times B)$$

is a local martingale, then ν_Y is the compensator of μ_Y . It follows that for any nonnegative predictable measurable function $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$,

$$\mathbb{E} \left[\int_s^t \int_{\mathbb{R}} g(u, x) \mu_Y(du, dx) \middle| \mathcal{F}(s) \right] = \mathbb{E} \left[\int_s^t \int_{\mathbb{R}} g(u, x) \nu_Y(du, dx) \middle| \mathcal{F}(s) \right]$$

provided the expectations are well-defined since Y is of finite variation. In essence, the compensator ν_Y governs the predictable (instantaneous) intensity of the jump structure encoded by μ_Y . Analogous to the representation of Y via its jump measure, one can define the compensator-driven finite variation process

$$A(t) = \int_0^t \int_{\mathbb{R}} x \nu_Y(du, dx)$$

such that the process $Y - A$ is a local martingale.

The compensator ν_Y characterizes the distributional structure of the process Y . Informally, for Borel sets $A \subset (0, T]$ and $B \subset \mathbb{R}$, $\nu_Y(A \times B)$ represents the expected number of jumps occurring in the time interval A with jump sizes in B . If $\nu_Y([0, t] \times \mathbb{R}) = \infty$, the process Y is said to exhibit infinite activity on $[0, t]$; otherwise, it is of finite activity. Analogous to classical analysis, if there exists a measurable function $\psi_Y : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that $\nu_Y(dt, dx) = \psi_Y(t, x) dx dt$, then ν_Y is said to be absolutely continuous with respect to Lebesgue measure, with density/intensity function ψ_Y . In most applied settings involving parametric stochastic processes, the compensator ν_Y is specified through a parameterized intensity function, which fully determines the statistical behavior of the jump structure.

It is instructive to illustrate the marked point process framework using the canonical example of a Poisson process with constant intensity, even though the general Lévy or jump-diffusion formulations already provide a sufficient theoretical foundation.

Example 3.1 (Poisson process: Compensator). Consider the Poisson process $N(t)$ with constant intensity $\beta > 0$ (“ λ ” is reserved for its later use as the market price of jump risk). The jump measure of N is

$$\mu_N(dt, dx) = \sum_{s: \Delta N(s)=1} \delta_{(s,1)}(dt, dx),$$

The compensator of μ_N is

$$\nu_N(dt, dx) = \beta \delta_1(dx) dt,$$

where δ_1 is the Dirac delta function centered at 1. Accordingly, the compensator process A_N is

$$A_N(t) = \int_0^t \int_{\mathbb{R}} x \nu_N(du, dx) = \beta \int_0^t \left(\int_{\mathbb{R}} x \delta_1(dx) \right) du = \beta t.$$

and the compensated process $N - A_N = (N(t) - \beta t)_{t \in [0, T]}$ is a martingale.

We next recall a Girsanov theorem for pure jump processes, following Theorem 3.24 of [31] (Chapter III, p. 172). To accommodate the marked point process representation and purely discontinuous dynamics of the time-changed fBm framework

considered in this paper, we specialize the result to jump measures by suppressing any continuous martingale component.

Lemma 3.2 (Girsanov theorem for pure jump processes, Theorem 3.24 of [31]). *Let Y be a local semimartingale, pure jump process with jump measure μ_Y and compensator ν_Y under the probability measure $\mathbb{P}$. Let $\lambda : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a predictable process satisfying*

$$\lambda(t, x) > 0, \quad \mathbb{P} \text{ a.s.} \quad \forall t \in [0, T], \quad x \in \mathbb{R},$$

and Novikov's integrability condition

$$\mathbb{E}^{\mathbb{P}} \left[\exp \left(\frac{1}{2} \int_0^T \int_{\mathbb{R}} (\lambda(s, x) - 1)^2 \nu_Y(ds, dx) \right) \right] < \infty.$$

Define the probability measure $\mathbb{Q} \ll \mathbb{P}$ by the Radon-Nikodym derivative process

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}(t)} = \exp \left\{ - \int_0^t \int_{\mathbb{R}} (\lambda(s, x) - 1) \nu_Y(ds, dx) + \int_0^t \int_{\mathbb{R}} \log \lambda(s, x) \mu_Y(ds, dx) \right\}. \quad (1)$$

Then, under $\mathbb{Q}$, the compensator of μ_Y is given by

$$\tilde{\nu}_Y(dt, dx) = \lambda(t, x) \nu_Y(dt, dx)$$

for all $t \in [0, T]$ and $x \in \mathbb{R}$, so that $Y - \int_0^t \int_{\mathbb{R}} x \tilde{\nu}_Y(ds, dx)$ is a $\mathbb{Q}$ -local-martingale.

In line with standard terminology, the predictable kernel λ is interpreted as the market price of jump risk. Continuing from Example 3.1, we illustrate the application of Girsanov's theorem in Lemma 3.2 to the constant-intensity Poisson process.

Example 3.3 (Poisson process: Girsanov theorem). Define the market price of risk

$$\lambda(t, x) = \frac{\tilde{\beta}}{\beta}$$

for all $t \in [0, T]$ and $x \in \mathbb{R}$, where $\tilde{\beta} > 0$ is some constant. Then, the Radon-Nikodym derivative is

$$\begin{aligned} \left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}(t)} &= \exp \left\{ - \int_0^t \int_{\mathbb{R}} \left(\frac{\tilde{\beta}}{\beta} - 1 \right) \nu_N(ds, dx) + \int_0^t \int_{\mathbb{R}} \log \left(\frac{\tilde{\beta}}{\beta} \right) \mu_N(ds, dx) \right\} \\ &= \exp \left\{ - \left(\tilde{\beta} - \beta \right) \int_0^t ds + N(t) \log \left(\frac{\tilde{\beta}}{\beta} \right) \right\} \\ &= \left(\frac{\tilde{\beta}}{\beta} \right)^{N(t)} e^{-t(\tilde{\beta} - \beta)} \end{aligned}$$

Under the probability measure $\mathbb{Q}$, N remains a Poisson process with compensator

$$\tilde{\nu}_N = \tilde{\beta} \delta_1(dx) dt$$

and the compensator process $\tilde{A}_N$ is

$$\tilde{A}_N(t) = \int_0^t \int_{\mathbb{R}} x \tilde{\nu}_N(ds, dx) = \int_0^t \tilde{\beta} ds = \tilde{\beta} t.$$

Similarly, the compensated process $N - \tilde{A}_N$ is a $\mathbb{Q}$ -martingale.

3.2. Gamma process. Let $\gamma = (\gamma(t; v))_{t \in [0, T]}$ denote a gamma process with unit mean rate and variance intensity $v > 0$, characterized by independent Gamma-distributed increments over non-overlapping intervals [40], and we adopt the convention that when $v = 0$, $\gamma(t; 0) = t$.²

The gamma process will subsequently serve as a stochastic time change, referred to as activity time, to distinguish it from calendar time. For any time $t \geq 0$ and lag $h > 0$, the increment $G_h := \gamma(t + h; v) - \gamma(t; v)$ follows a Gamma distribution with shape parameter h/v and rate parameter $1/v$, or equivalently, the probability density function of G_h is given by

$$f_{G_h}(g) = \frac{g^{h/v-1} e^{-g/v}}{\Gamma(h/v) v^{h/v}}, \quad \forall g > 0, h > 0 \quad (2)$$

where $\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz$ denotes the gamma function for all $x > 0$. The increments have expectation $E[G_h] = h$ and variance $\text{Var}[G_h] = vh$. Intuitively, when viewed as a stochastic time change (or subordinator), the gamma activity time evolves at unit speed in expectation relative to calendar time. The volatility parameter v governs the variability of the activity clock around its mean and therefore controls the randomness in the accumulation of activity time per unit of calendar time.

The gamma process γ is a non-decreasing Lévy process with independent and stationary increments, and is therefore referred to as a subordinator. Moreover, it is an infinite-activity (infinitely many jumps on any finite interval), pure jump (purely discontinuous) process. The gamma process admits the following marked point process representation³ with absolutely continuous compensator ν_γ satisfying

$$\nu_\gamma(dt, dg) = \psi_\gamma(g) dg dt, \quad \forall g > 0, t \in [0, T],$$

where the compensator density function is given by

$$\psi_\gamma(g) = \frac{e^{-g/v}}{vg}, \quad \forall g > 0. \quad (3)$$

3.3. The fBm process. The standard (one-sided) fBm $B_H = (B_H(t))_{t \in [0, T]}$ is a centered continuous Gaussian process with stationary increments, initial condition $B_H(0) = 0$, and covariance function

$$E[B_H(t)B_H(s)] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad \forall s, t \in [0, T],$$

where $H \in (0, 1)$ denotes the Hurst coefficient. The process is H -self-similar, satisfying $B_H(ct) \stackrel{d}{=} |c|^H B_H(t)$ for all $c > 0$, and possesses sample paths that are almost surely α -Hölder continuous for any order $\alpha \in [0, H)$. When $H = 0.5$, the fBm reverts to the classical Brownian motion.

Distinct from classical Brownian motion, the fBm exhibits dependent increments. When $H > 0.5$, B_H admits smoother sample paths than standard Brownian motion,

²It is straightforward to generalize the specification by introducing an additional parameter that allows the expected activity rate to differ from calendar time. However, such a generalization lacks a clear economic motivation. Accordingly, we normalize the mean activity rate to unity.

³While the Lévy measure is often sufficient in the Lévy process setting, we adopt the marked point process formulation because it extends naturally to the subsequent analysis. In particular, the time-changed fBm introduced later is not a Lévy process due to the loss of independent increments. The compensator-based representation therefore provides the appropriate framework for characterizing its jump structure.

and its increments possess positive serial correlation and display long-range dependence (LRD), characterized by the hyperbolic rate of decay in autocorrelations such that distant observations retain non-negligible influence on future dynamics.⁴ This contrasts sharply with classical Brownian motion, whose increments are independent and Markovian, and thus exhibit no temporal persistence. Such dependence structures are particularly relevant for modeling persistent serial correlation. In contrast, when $H < 0.5$, the increments are negatively autocorrelated, and the process exhibits comparatively rough sample paths. Crucially, when $H \neq 0.5$, the fBm process ceases to be a semimartingale, which precludes the direct application of standard Itô calculus and the risk-neutral valuation framework.

Furthermore, the fBm exhibits anomalous diffusion, whereby its variance scales nonlinearly over time, $\text{Var}[B_H(t)] \propto t^{2H}$, deviating from the linear time scaling of standard Brownian motion. Accordingly, the process is superdiffusive for $H > 0.5$, with variance growing faster than linearly in time, and subdiffusive for $H < 0.5$, with slower-than-linear growth. In addition, the Hurst parameter governs path regularity; smaller values of H correspond to rougher trajectories with weaker Hölder continuity, while larger values of H yield progressively smoother sample paths.

A particularly useful characterization of fractional Brownian motion is its Volterra integral representation in terms of an underlying Brownian motion. Specifically,

$$B_H(t) = \int_0^t K_H(t, u) dB(u), \quad \forall t \in [0, T], \quad (4)$$

where $B = (B(t))_{t \in [0, T]}$ is a standard Brownian motion and K_H is a deterministic kernel given by

$$K_H(t, u) = \sigma_H \frac{(t-u)^{H-0.5}}{\Gamma(H+\frac{1}{2})} {}_2F_1\left(0.5-H; H+0.5, H+\frac{3}{2}; 1-\frac{t}{u}\right) 1_{[0,t)}(u) \quad (5)$$

$$\sigma_H^2 = \frac{2\pi H(H-\frac{1}{2})}{\Gamma(2-2H)\sin(\pi(H-\frac{1}{2}))} \quad (6)$$

for all $t, u \in [0, T]$, where ${}_2F_1$ is the Gaussian hypergeometric function. The filtration generated by B_H coincides with that generated by the driving Brownian motion B , $\sigma((B_H(u))_{u \in [0, t]}) = \sigma((B(u))_{u \in [0, t]})$ for all $t \in [0, T]$. For further technical details on fBm, we refer the readers to [44, 54, 12].

In parallel with the classical Brownian motion, a version of Girsanov's theorem holds for fBm.

Lemma 3.4 (Girsanov Theorem for fBm, [12]). *Let $(\lambda(t))_{t \in [0, T]}$ be a square-integrable, adapted process with respect to Lebesgue measure such that $\mathbb{E}^\mathbb{P}[\mathcal{L}(T)] = 1$, where*

$$\mathcal{L}(t) = \exp\left\{-\int_0^t \lambda(v) dB(v) - \frac{1}{2} \int_0^t \lambda(v)^2 dv\right\}, \quad \forall t \in [0, T]. \quad (7)$$

⁴Several notions of LRD are used in the literature. A standard definition characterizes a covariance-stationary process $\{X_k\}_{k \in \mathbb{Z}}$ as exhibiting LRD if its autocovariance function $\{\gamma_X(k)\}_{k \in \mathbb{Z}}$ is not absolutely summable, i.e., $\sum_{k=-\infty}^{\infty} |\gamma_X(k)| = \infty$. Alternatively, LRD can be characterized by the asymptotic rate of decay of the autocorrelation function, satisfying $\gamma_X(k) \sim k^{2H-2}$ as $k \rightarrow \infty$, or, in the frequency domain, via the divergence of the spectral density at the origin, with the spectral density behaving as $\lambda^{-(2H-1)}$ as $\lambda \rightarrow 0$ for $H \in (0.5, 1)$. The increments of the fBm process satisfy all these definitions when $H > 0.5$.

Define the equivalent probability measure $\mathbb{Q}$ by the Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}(t)} = \mathcal{L}(t)$. Then, $(B^H(t) + \int_0^t K_H(t, u)\lambda(u) du)_{t \in [0, T]}$ under $\mathbb{P}$ has the same law as $(B^H(t))_{t \in [0, T]}$ under $\mathbb{Q}$.

Owing to its analytical tractability and ability to capture both persistence and roughness, fBm has been widely adopted across disciplines, including hydrology [29, 45], climate science [34, 30, 48, 20, 57], operations [1, 23, 47, 37], and life sciences [59, 25, 49]. In finance, fBm has been extensively studied in relation to long memory in asset returns [22, 65, 28] and the roughness and clustering of stochastic volatility [64, 51]. In actuarial science and insurance, it has been used to model temperature and mortality dynamics for pricing weather derivatives and life-contingent claims [60, 63, 69].

We now introduce notation that will be used in the construction of the time-changed fBm in the following section. Let $\phi(\cdot; t, g)$ denote the conditional probability density function (PDF) of the increment $B_H(t + g) - B_H(t)$ given $\mathcal{F}(t)$, for $t \in [0, T]$ and lag $g \geq 0$,

$$\phi(x; t, g) = \frac{1}{\sqrt{2\pi}\sigma_B(g; t)} \exp \left\{ -\frac{1}{2} \left(\frac{x - m_B(g; t)}{\sigma_B(g; t)} \right)^2 \right\}, \quad \forall x \in \mathbb{R}, \quad (8)$$

where $m_B(g; t)$ and $\sigma_B^2(g; t)$ are the conditional mean and variance of the increment $B_H(t + g) - B_H(t)$,

$$\begin{aligned} m_B(g; t) &= \mathbb{E}[B_H(t + g) - B_H(t) | \mathcal{F}(t)] \\ &= \int_0^t K_H(t + g, u) - K_H(t, u) dB(u) \end{aligned} \quad (9)$$

$$\begin{aligned} \sigma_B^2(g; t) &= \text{Var}[B_H(t + g) - B_H(t) | \mathcal{F}(t)] \\ &= \int_t^{t+g} K_H(t + g, u)^2 du. \end{aligned} \quad (10)$$

4. Time-changed fBm process. This section introduces the time-changed fBm X underlying the fVG model. The process $X = (X(t))_{t \in [0, T]}$ is obtained by evaluating fBm B_H at a gamma time change γ , where B_H and γ are assumed to be independent. Specifically,

$$X(t) \equiv X(t; H, v) = B_H(\gamma(t; v)), \quad \forall t \in [0, T], v \geq 0, H \in (0, 1).$$

4.1. Properties. We begin by summarizing several key properties of the time-changed fBm process established in the literature [35, 36]. Since the case $H = 0.5$ reduces to the classical Variance Gamma model, which has been extensively studied in the literature, we restrict attention to $H \in (0, 1) \setminus \{0.5\}$ in the subsequent analysis for simplicity. First, the probability density function (PDF) of the time-changed fBm process X is derived by marginalizing the Gaussian conditional PDF of the fBm over the Gamma-distributed PDF governing the increments of the activity time process,

$$\begin{aligned} f_{X(t)}(x) &= \int_0^\infty \phi(x; 0, g) f_{G_t}(g) dg \\ &= \frac{1}{\sqrt{2\pi} v^{t/v} \Gamma(t/v)} \int_0^\infty g^{t/v - H - 1} e^{-x^2 g^{-2H}/2 - g/v} dg \end{aligned} \quad (11)$$

for all $x \in \mathbb{R} \setminus \{0\}$, where f_{G_t} and ϕ are defined in Equations (2) and (8), respectively. Second, the covariance function of X takes the form

$$\mathbb{E}[X(s)X(t)] = \frac{v^{2H}}{2} \left(\frac{\Gamma(\frac{t}{v} + 2H)}{\Gamma(\frac{t}{v})} + \frac{\Gamma(\frac{s}{v} + 2H)}{\Gamma(\frac{s}{v})} - \frac{\Gamma(\frac{|t-s|}{v} + 2H)}{\Gamma(\frac{|t-s|}{v})} \right)$$

for all $s, t \in [0, T]$. Importantly, the gamma activity time change preserves the long-memory properties of the underlying fBm. The covariance function of the non-overlapping increments of X is

$$\begin{aligned} & \mathbb{E}[(X(t + (n+1)h) - X(t + nh))(X(t + h) - X(t))] \\ &= \frac{v^{2H}}{2} \left(\frac{\Gamma(\frac{(n-1)h}{v} + 2H)}{\Gamma(\frac{(n-1)h}{v})} - 2 \frac{\Gamma(\frac{nh}{v} + 2H)}{\Gamma(\frac{nh}{v})} + \frac{\Gamma(\frac{(n+1)h}{v} + 2H)}{\Gamma(\frac{(n+1)h}{v})} \right) \end{aligned} \quad (12)$$

for all $n \in \mathbb{N}, h > 0$, and behaves asymptotically as $H(2H - 1)h^{2H}n^{2H-2}$ as $n \rightarrow \infty$. Consequently, the increment process exhibits long-range dependence (LRD) whenever $H > 0.5$. The time-changed fBm process also admits the characteristic function

$$\begin{aligned} \varphi_{X(t)}(u) &= \mathbb{E}[e^{iuX(t)}] = \mathbb{E}\left[e^{-\frac{u^2}{2}\gamma(t)^{2H}}\right] \\ &= \frac{(1/v)^{t/v}}{\Gamma(t/v)} \int_0^\infty r^{t/v-1} \exp\left\{-\frac{r}{v} - \frac{u^2}{2}r^{2H}\right\} dr \\ &= \begin{cases} \frac{1}{\Gamma(t/v)} \sum_{n=0}^\infty \frac{\Gamma(t/v+2Hn)}{n!} \left(-\frac{u^2 v^{2H}}{2}\right)^n, & H < 0.5 \\ \frac{1}{\Gamma(t/v)} \mathbf{H}_{1,1}^{1,1} \left(\frac{u^2 v^{2H}}{2} \middle| \begin{matrix} (1-t/v, 2H) \\ (0, 1) \end{matrix} \right), & H > 0.5 \end{cases}. \end{aligned}$$

The final equality follows from the Wright series representation [67] when $H < 0.5$ and Fox- $\mathbf{H}$ representation [19] when $H > 0.5$. The characteristic function provides an alternative characterization of the marginal distribution of the time-changed fBm process and may facilitate the derivation of further distributional and pricing-related properties. We leave a more detailed investigation of these applications for future research.

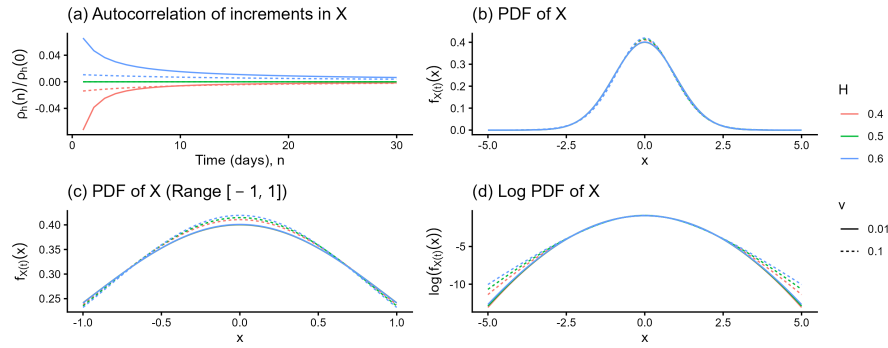


FIGURE 1. Autocorrelation of the increments in X and PDF of X across various v, H

Figure 1(a) plots the autocorrelation function $\rho_h(n)/\rho_h(0)$ for various combinations of H and v up to lag $n = 30$, where $h = 1/252$ corresponds to the average length of a trading day expressed in years and $\rho_h(n)$ is given by Equation (12). The autocorrelation profiles indicate the presence of LRD in the increments of X when $H > 0.5$, as demonstrated by the slow decay of the autocorrelation function. In addition, increasing the variance intensity parameter v reduces the magnitude of the autocorrelations, except when n is large and $H < 0.5$.

Figure 1(b) shows the PDF of $X(1)$, where $f_{X(t)}(x)$ is given in Equation (11). Since the overall scale may obscure important features of the distribution, two supplementary visualizations are provided. Figure 1(c) displays the same PDF with the domain restricted to $x \in [-1, 1]$, thereby emphasizing the behavior around the peak, while Figure 1(d) presents the logarithm of the PDF to better illustrate the tail behavior. The results show that larger values of H and v lead to a lower peak and heavier tails, indicating an increase in dispersion and tail thickness. However, when v is small, variations in H have only a limited effect on the shape of the PDF. This suggests that v is the primary parameter governing the kurtosis of the time-changed fBm process.

Next, we derive the unconditional moments of the time-changed fBm process in Proposition 4.1.

Proposition 4.1 (Unconditional raw moments of X). *For any $n \in \mathbb{N}$,*

$$\mathbb{E}[B_H(\gamma(t))^n] = \begin{cases} v^{nH} \frac{\Gamma(t/v+nH)}{\Gamma(t/v)} (n-1)!!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}. \quad (13)$$

It follows from Proposition 4.1 that the unconditional mean and skewness of X are both zero. Moreover, the unconditional variance is given by $\text{Var}[X(t)] = v^{2H} \frac{\Gamma(t/v+2H)}{\Gamma(t/v)}$, and the unconditional kurtosis is

$$\begin{aligned} \frac{\mathbb{E}[(X(t) - \mathbb{E}[X(t)])^4]}{\text{Var}[X(t)]^2} &= \frac{3v^{4H}\Gamma(t/v+4H)/\Gamma(t/v)}{(v^{2H}\Gamma(t/v+2H)/\Gamma(t/v))^2} \\ &= 3 \frac{\Gamma(t/v+4H)\Gamma(t/v)}{\Gamma(t/v+2H)^2} \\ &= \frac{3}{{}_2F_1(-2H, 2H; t/v+2H; 1)} \\ &= 3 \prod_{k=0}^{\infty} \frac{(t/v+2H+k)^2}{(t/v+4H+k)(t/v+k)}, \quad \forall t \geq 0, v > 0, \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function. Thus, X is leptokurtic, and the variance intensity parameter v controls the kurtosis of X . As $t \rightarrow \infty$, the kurtosis decreases and converges to 3, corresponding to the kurtosis of a Gaussian process. These results are consistent with the observations shown in Figure 1 and with the findings of [35] and, in the special case $H = 0.5$, of [40].

Next, we describe the simulation procedure for the time-changed fBm X on the interval $[0, T]$ with time step $a > 0$, given its parameters $v \geq 0$ and $H \in (0, 1)$. The idea is to first simulate both the discretized fBm and the gamma processes independently, and then evaluate the fBm at the corresponding realized activity

times generated by the gamma process. To ensure numerical accuracy, the fBm is generated on a finer (secondary) grid than that used for the gamma activity time (primary grid).

Let the step sizes of the primary and secondary grids be a and b , respectively, with $a > b > 0$. The primary grid $\{t_n\}_{n=1}^N$ is constructed by $t_n = na$, where $N = \lfloor T/a \rfloor$ denotes the number of primary grid points. We first simulate the gamma process $\{\hat{\gamma}_n\}_{n=1}^N$ with $\hat{\gamma}_n = \gamma(t_n)$. Subsequently, we construct the secondary grid $\{s_j\}_{j=0}^J$ via $s_j = jb$, where $J = \lfloor \hat{\gamma}_N/b \rfloor$ represents the number of secondary grid points. Next, we generate a sample path of the fBm $\{\hat{B}_{H,j}\}_{j=0}^J$ on the secondary grid, with $\hat{B}_{H,j} = B_H(s_j)$. We then map the realized activity times onto the fBm path by selecting $\hat{B}_{H,n} = \hat{B}_{H,j}$ with $j = \lfloor \hat{\gamma}_n/b \rfloor$ for each $n \in \{1, \dots, N\}$. The resulting sequence $\{\hat{B}_{H,n}\}_{n=1}^N$ constitutes a single simulated path of the time-changed fBm process X . Throughout the paper, we set $a = 1/252$, corresponding to the length of a trading day measured in years, and $b = a/100$.

For simulation of the gamma process, we draw independent and identically distributed gamma increments with shape parameter a/v and rate parameter $1/v$, and construct the process via cumulative summation. The fBm paths are generated by summing simulated increments obtained through the Wood–Chan method, which exploits a circulant embedding of the covariance matrix of fractional Gaussian noise and employs the Fast Fourier Transform for computational efficiency [66].

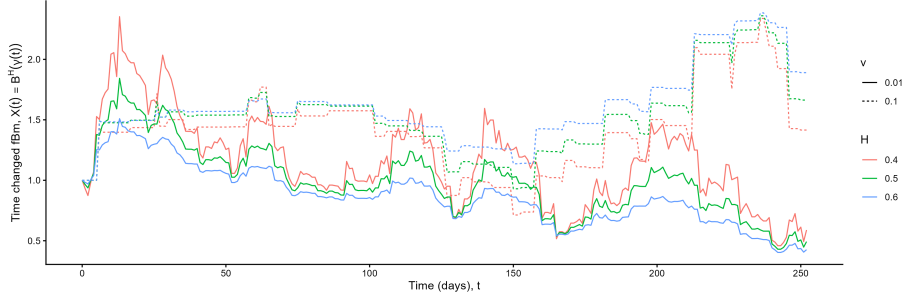


FIGURE 2. Simulated sample paths of X under various v, H

Figure 2 presents simulated sample paths of time-changed fBm on the interval $[0, 1]$ under various combinations of the parameters $\{v, H\}$. To facilitate direct comparison across parameter settings, the same random seed is used in all simulations, thereby reducing sampling variability in comparisons and isolating the effects of v and H on the sample paths. When $H < 0.5$, the sample paths exhibit a relatively rough and irregular appearance, whereas for $H > 0.5$, the trajectories become noticeably smoother. Thus, Hurst exponent governs the visual irregularity of the sample paths, a property inherited from the underlying fBm. Note that the gamma subordination fundamentally changes the path regularity of fBm, and the resulting X is pure jump and of bounded variation.

In contrast, as v increases, the trajectories appear increasingly intermittent, characterized by extended periods of little or no movement followed by abrupt changes. This phenomenon arises because a larger variance intensity parameter induces greater variability in the increments of the gamma activity time. Although

the mean rate of the gamma activity time remains unchanged, its increment distribution becomes more positively skewed and heavy-tailed, resulting in more frequent occurrences of both very small and exceptionally large increments. Consequently, the time-changed process exhibits prolonged flat segments interspersed with sudden, pronounced movements. Overall, the results indicate that the Hurst exponent H primarily controls the irregularity and scaling of jump magnitudes inherited from the underlying fBm, while the variance intensity parameter v primarily governs the intermittency and irregular timing induced by the stochastic clock.

4.2. Marked point representation. To characterize the stochastic drift of the time-changed fBm process X , it is more convenient to represent X as a marked point process. Since fBm has continuous sample paths and the gamma activity time is an infinite activity, pure jump process, the resulting time-changed process X is also a purely discontinuous, pure jump process.

Let ν_X denote the compensator of X and let ψ_X be the associated density of ν_X , satisfying

$$\nu_X(dt, dx) = \psi_X(t, x) dx dt, \quad \forall t \in [0, T], x \in \mathbb{R}. \quad (14)$$

By the construction of X , the compensator density ψ_X can be expressed as the convolution of the conditional density of the fractional Brownian motion increments with the density of the gamma activity process,

$$\begin{aligned} \psi_X(t, x) &= \int_0^\infty \phi(x; \gamma(t; v), g) \psi_\gamma(g) dg \\ &= \int_0^\infty \frac{\exp \left\{ -\frac{1}{2} \left(\frac{x - m_B(g; \gamma(t; v))}{\sigma_B(g; \gamma(t; v))} \right)^2 \right\}}{\sqrt{2\pi} \sigma_B(g; \gamma(t; v))} \cdot \frac{1}{vg} e^{-\frac{g}{v}} dg, \quad \forall t \in [0, T], x \in \mathbb{R}, \end{aligned} \quad (15)$$

where the functions ψ_γ , ϕ , m_B and σ_B are defined in Equations (3), (8), (9), and (10), respectively. The conditional moments m_B and σ_B are evaluated at the latent gamma activity time $\gamma(t)$,

$$\begin{aligned} m_B(g; \gamma(t)) &= \mathbb{E}[B_H(\gamma(t) + g) - B_H(\gamma(t)) | \mathcal{F}(t-)] \\ &= \int_0^{\gamma(t)} K_H(\gamma(t) + g, u) - K_H(\gamma(t), u) dB(u) \\ \sigma_B^2(g; \gamma(t)) &= \text{Var}[B_H(\gamma(t) + g) - B_H(\gamma(t)) | \mathcal{F}(t-)] \\ &= \int_{\gamma(t)}^{\gamma(t)+g} K_H(\gamma(t) + g, u)^2 du \end{aligned}$$

for all $g \geq 0$ and $t \in [0, T]$.

4.3. Semimartingale behavior and equivalent probability measure changes. This section presents the main theoretical results establishing the suitability of the time-changed fBm X as a driving process for asset price models within the classical arbitrage-free pricing framework. We first prove that the time-changed fBm is a semimartingale and then characterize its compensator.

Theorem 4.2 (Semimartingale X). *The time-changed fBm $X = B_H(\gamma)$ is a semimartingale.*

Proposition 4.3 (Compensator of X). *There exists a unique, continuous, predictable, finite variation process $A_X = \int_0^t a_X(s) ds$ such that $X - A_X$ is a $\mathbb{P}$ -martingale, where*

$$a_X(s) = \int_{\mathbb{R}} x \psi_X(s, x) dx = \int_0^\infty m_B(g; \gamma(s)) \psi_\gamma(g) dg, \quad \forall s \in [0, t].$$

Theorem 4.2 establishes that the gamma time-changed fBm X is a semimartingale, while Proposition 4.3 provides an explicit characterization of its predictable compensator. In particular, the compensator is unique, and the compensated process $X - A_X$ is a true $\mathbb{P}$ -martingale. Intuitively, the process $a_X = (a_X(t))_{t \in [0, T]}$ represents the instantaneous predictable drift of the time-changed fBm, in the sense that $a(t) \approx \mathbb{E}[B_H(\gamma(t+dt)) - B_H(\gamma(t)) | \mathcal{F}(t-)] / dt$, which quantifies the conditional expected infinitesimal rate of change given the available information. For the special case $H = 0.5$, the underlying fBm reduces to a standard Brownian motion, whose increments have zero conditional mean. In this case, $a_X(t) = 0$ for all $t \in [0, T]$, and therefore, the compensator process vanishes, $A_X(t) = 0$, and the time-changed Brownian motion is a martingale.

Consequently, as discussed in the introduction, X may be employed as the driving noise process in asset price models within the classical continuous-time, continuous-trading, arbitrage-free pricing framework, thereby overcoming the principal obstacle to the direct use of fBm, namely its failure to be a semimartingale. Moreover, the existence of a unique predictable compensator facilitates the derivation of sufficient conditions on the drift of an asset price process under which an equivalent change of probability measure renders the discounted price process a martingale. The following corollary establishes Girsanov's change-of-measure theorem for the time-changed fBm.

Corollary 4.4 (Girsanov theorem for time-changed fBm). *Consider a time-changed fBm process $X = B_H(\gamma)$ with compensator ν_X under the statistical measure $\mathbb{P}$. Suppose there exists a predictable process $\lambda > 0$ satisfying all the necessary conditions in Lemma 3.2, and define the probability measure $\mathbb{Q}$ by the Radon–Nikodym derivative in Equation (1). Then, there exists a unique, predictable, finite-variation compensator process $\tilde{A}_X(t) = \int_0^t \int_{\mathbb{R}} x \lambda(s, x) \nu_X(ds, dx)$ such that the compensated process $X - \tilde{A}_X$ is a $\mathbb{Q}$ -local-martingale. Furthermore, if*

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_{\mathbb{R}} x^2 \lambda(s, x) \psi_X(s, x) dx ds \right] < \infty, \quad (16)$$

then $X - \tilde{A}_X$ is a $\mathbb{Q}$ -martingale.

Corollary 4.4 characterizes the predictable compensator of the time-changed fBm under an equivalent risk-neutral measure. A sufficient condition for Equation (16) is that the market price of risk be uniformly bounded, that is, $|\lambda(t, x)| < M$ for some constant $M > 0$. Indeed, since the compensator under $\mathbb{Q}$ satisfies $\tilde{\nu} = \lambda \nu$, and the expected quadratic variation of X under $\mathbb{P}$ is finite, it follows that

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_{\mathbb{R}} x^2 \lambda(s, x) \psi_X(s, x) dx ds \right] < M \mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_{\mathbb{R}} x^2 \psi_X(s, x) dx ds \right] < \infty.$$

In the general case, with an arbitrary market price of jump risk $\lambda(x, t)$, the distribution of $B_H(\gamma)$ under $\mathbb{Q}$ is not generally known. The Girsanov transformation

establishes an equality in distribution between the appropriately drift-adjusted processes under $\mathbb{P}$ and $\mathbb{Q}$. This equality does not, in general, imply that $B_H(\gamma)$ retains the structural representation of a gamma-time-changed fBm under $\mathbb{Q}$, unless the change of measure preserves both the gamma structure of the time change and its independence from the transformed fBm.

Finally, we highlight some challenges arising in the application of time-changed fBm within financial modeling. In particular, the conditional distribution of $X(t)$ given $\mathcal{F}(s)$, for $s \leq t$, cannot be determined solely based on the realizations of X up to time s . Although its conditional law admits a representation in terms of the latent gamma activity time $\gamma(s)$, the realized values of $\gamma(s)$ are unobservable. Specifically, under the natural filtration $\mathcal{F}_X(s) = \sigma(\{X(u) : u \in [0, s]\})$, the activity time $\gamma(s)$ is not $\mathcal{F}_X(s)$ -measurable. Consequently, the constituent processes B_H and γ cannot be separately identified from the observations of $X = B_H(\gamma)$. Likewise, the compensator process A_X is not adapted to the filtration generated by X since it depends on the conditional moments $m_B(g, \gamma(s))$ and $\sigma_B^2(g; \gamma(s))$, which are functions of the latent activity time γ and therefore cannot be evaluated directly from the observed sample path of X .

Although the conditional distribution of X is unavailable in closed form, it can be approximated by simulation. The Gaussian–Gamma mixture representation of the time-changed fBm makes unconditional path simulation of X computationally straightforward. The conditional distribution can then be approximated by retaining only those simulated paths that are sufficiently close to the observed history. Specifically, given historical observations $\{\hat{X}(t_k)\}_{k=0}^K$ with $0 = t_0 < t_1 < \dots < t_K = s$ and parameters $\{v, H\}$, the conditional distribution may be approximated by

$$\hat{F}_{X(t)|\mathcal{F}(s)}(x) = \frac{\sum_{i=1}^N \mathbf{1}[X^{(i)}(t) \leq x] \mathbf{1}[|X^{(i)}(t_k) - \hat{X}(t_k)| < \delta_k \quad \forall k]}{\sum_{i=1}^N \mathbf{1}[|X^{(i)}(t_k) - \hat{X}(t_k)| < \delta_k \quad \forall k]}$$

which converges to the true conditional distribution as $N \rightarrow \infty$ and $\delta_k \rightarrow 0$ for all k on a refining observation grid with appropriate regularity assumptions. Simulation of X is discussed in Section 4.1, while parameter estimation is presented in Section 6. Conditional quantities of practical interest, such as forecasts based on conditional expectations and measures of future uncertainty based on conditional variances, can then be evaluated directly from the simulated distribution.

4.4. Skewness. The standard time-changed fBm X is symmetric and therefore cannot capture skewness that is commonly observed in financial return series. In this section, we introduce an extension analogous to the VG model of [40] by appending an additional drift term $\theta\gamma(t; v)$ to introduce asymmetry, thereby enabling the model to capture skewed return distributions.

Consider the process $W = (W(t))_{t \in [0, T]}$, where

$$W(t) \equiv W(t; \theta, \sigma, v, H) = \theta\gamma(t; v) + \sigma X(t; H, v).$$

The four parameters in W have distinct distributional effects. The parameter $\theta \in \mathbb{R}$ primarily controls the mean and skewness of the increments, $v \geq 0$ controls the variability of the activity time and hence the excess kurtosis, $\sigma \geq 0$ determines the overall scale of the process, and $H \in (0, 1)$ governs the path regularity, degree of persistence, and the rate at which moments scale with the time horizon. This separation of roles is useful for estimation, as changes in skewness, kurtosis, scale,

and temporal dependence can be attributed to different parameters of the model. The proposed specification represents a fractional extension of the classical VG model. When $H = 0.5$, the model coincides with the VG process. For $H \neq 0.5$, the model retains the variance-gamma mechanism responsible for skewness and excess kurtosis while additionally incorporating long-range dependence and fractional scaling through the Hurst parameter.

The distribution of W admits a normal-gamma mixture representation. Its unconditional density can therefore be obtained by integrating the conditional Gaussian density with respect to the distribution of γ . We begin by deriving the unconditional moments of the increments in W .

Proposition 4.5 (Unconditional moments of $W(t)$). *For any $n \in \mathbb{N}$, the n -th unconditional raw moment of the increments in W is*

$$\begin{aligned} & \mathbb{E}[(W(t+h) - W(t))^n] \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k! (n-2k)!} \theta^{n-2k} \sigma^{2k} v^{n-2k(1-H)} \frac{\Gamma\left(\frac{h}{v} + n - 2k(1-H)\right)}{\Gamma\left(\frac{h}{v}\right)}, \end{aligned} \quad (17)$$

and the n -th unconditional central moment of the increments in W is

$$\begin{aligned} & \mathbb{E}[(W(t+h) - W(t) - \mathbb{E}[W(t+h) - W(t)])^n] \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k! (n-2k)!} \theta^{n-2k} \sigma^{2k} \sum_{\ell=0}^{n-2k} \binom{n-2k}{\ell} (-h)^{n-2k-\ell} v^{2kH+\ell} \frac{\Gamma\left(\frac{h}{v} + 2kH + \ell\right)}{\Gamma\left(\frac{h}{v}\right)}. \end{aligned} \quad (18)$$

Following Proposition 4.5, explicit expressions for the first four moments are presented below. These expressions are subsequently used in Section 6 for parameter estimation.

Corollary 4.6. *For any $t \geq 0$ and $h \geq 0$, the unconditional expectation and the second through fourth unconditional central moments of the increments in W are*

$$\begin{aligned} & \mathbb{E}[W(t+h) - W(t)] = \theta h \\ & \mathbb{E}[(W(t+h) - W(t) - \mathbb{E}[W(t+h) - W(t)])^2] = \theta^2 v h + \sigma^2 v^{2H} \frac{\Gamma\left(\frac{h}{v} + 2H\right)}{\Gamma\left(\frac{h}{v}\right)} \\ & \mathbb{E}[(W(t+h) - W(t) - \mathbb{E}[W(t+h) - W(t)])^3] = 2\theta^3 v^2 h \\ & \quad + 6H\sigma^2 \theta v^{1+2H} \frac{\Gamma\left(\frac{h}{v} + 2H\right)}{\Gamma\left(\frac{h}{v}\right)} \\ & \mathbb{E}[(W(t+h) - W(t) - \mathbb{E}[W(t+h) - W(t)])^4] = \theta^4 (3h^2 v^2 + 6h v^3) \\ & \quad + 6\sigma^2 \theta^2 v^{2H+1} (h + 2H(2H+1)v) \frac{\Gamma\left(\frac{h}{v} + 2H\right)}{\Gamma\left(\frac{h}{v}\right)} + 3\sigma^4 v^{4H} \frac{\Gamma\left(\frac{h}{v} + 4H\right)}{\Gamma\left(\frac{h}{v}\right)}. \end{aligned}$$

Figure 3 illustrates the moments reported in Corollary 4.6 for time lags $h = n/252$, where $n \in \{1, \dots, 30\}$. Under this convention, n may be interpreted as the number of trading days. Throughout, we fix $\sigma = 0.2$, as this parameter primarily

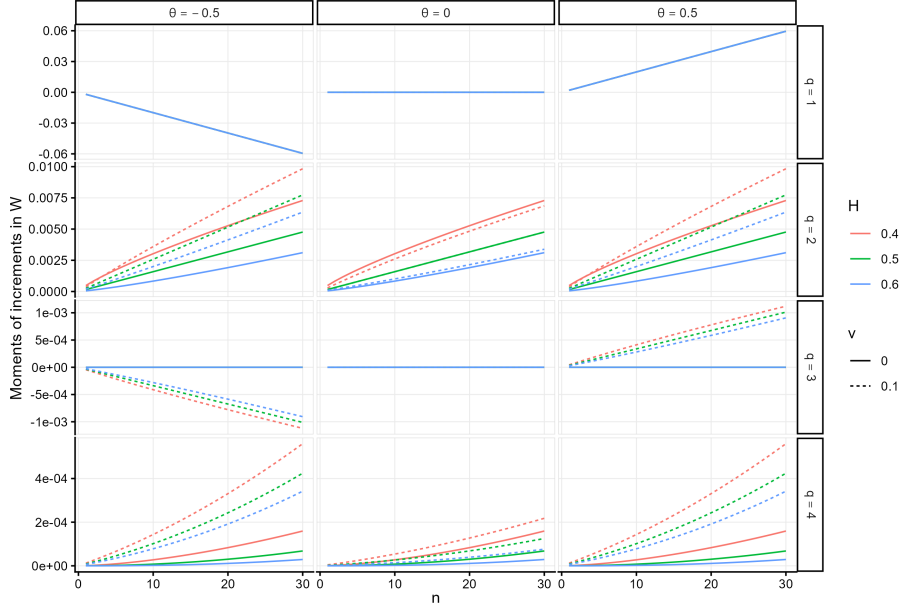


FIGURE 3. Moments of the increments in W given in Corollary 4.6 across parameters $\{H, v, \theta\}$.

affects the scale of W and does not materially alter the qualitative behavior of its moments.

When $\theta = 0$, both the first raw moment and the third central moment remain identically zero, indicating that θ is the primary driver of the mean and skewness of W . As θ increases, the third central moment becomes more pronounced, with the effect being particularly strong for smaller values of H . The Hurst parameter H also influences the rate at which higher-order moments evolve with the time lag h . In particular, using the asymptotic relation $\Gamma(h/v + 2kH)/\Gamma(h/v) \sim (h/v)^{2kH}$ as $h \rightarrow \infty$, the central moments of W grow according to a power law depending on the value of H .

Similar to Section 4.1, the parameter v governs the tail heaviness, or kurtosis, of W . When $\theta = 0$, the kurtosis is monotonically increasing in v and converges to 3 as $v \rightarrow 0$. Since v governs the variability of the activity-time process, larger values of v correspond to greater fluctuations in the gamma activity clock, which in turn induces larger fluctuations in the conditional variance and produces heavier tails of W .

Overall, Figure 3 demonstrates that W provides separate control over the key distributional features of financial returns, allowing it to simultaneously capture skewness, heavy tails, and long-range dependence, which are three prominent features of financial return data.

5. The fractional variance gamma (fVG) model. In this section, we use the preceding results concerning time-changed fBm to construct the fVG model within the standard continuous-time, continuous-trading arbitrage-free option pricing framework. The key idea is that the randomness in an asset price process,

represented by a generic pure jump innovation process Y , is generated by a time-changed fBm, $X = B_H(\gamma)$, where the fBm B_H is evaluated at gamma activity time γ .

The adoption of activity time has a long tradition in option pricing and has a clear economic interpretation; see, e.g., [40, 21, 42]. Activity time corresponds to the times at which trades are executed and therefore reflects the accumulation of trading activity. During periods of high trading intensity, prices are updated more frequently and typically exhibit greater volatility. By contrast, between trading times, there are no transactions and hence no observable changes in prices. Consequently, for option valuation it is natural to model the price process in activity time, which reflects the flow of trades, rather than calendar time, which evolves exogenously at a constant rate.

5.1. The setup. Consider a frictionless and competitive market consisting of a stock and a default-free money market account (MMA) that trade continuously over the time horizon $[0, T]$. Without loss of generality, we assume that the stock pays no cash flows and that the risk-free spot rate process $r = (r(t))_{t \in [0, T]}$ is deterministic. Dividend payments can be incorporated at the cost of additional notation, and stochastic interest-rate dynamics can be introduced through standard extensions; neither affects the key contribution of the paper.

The fractional Variance Gamma (fVG) model is characterized by the stock price process

$$S(t) = S(0) \exp \{ \Xi(t) + Y(t) \}, \quad (19)$$

where $S(0) > 0$ and $\Xi = (\Xi(t))_{t \in [0, T]}$ denotes a predictable drift process. For convenience, we assume that Ξ is absolutely continuous, so that

$$\Xi(t) = \int_0^t \xi(s) ds$$

for all $t \in [0, T]$, where ξ is a predictable, locally integrable process, with $\xi(t) > r(t)$ for all $t \in [0, T]$. Accordingly, Ξ is continuous with finite variation. The process $Y = (Y(t))_{t \in [0, T]}$ represents the stock's innovation process and is modeled as a pure jump process with jump measure μ_Y . Let ν_Y denote its compensator with density ψ_Y under the statistical measure $\mathbb{P}$. Following Section 3.1, the processes Y and its compensator process A_Y under $\mathbb{P}$ admit the representations

$$Y(t) = \int_0^t \int_{\mathbb{R}} x \mu_Y(ds, dx), \quad A_Y(t) = \int_0^t \int_{\mathbb{R}} x \nu_Y(ds, dx) \quad \forall t \in [0, T].$$

Although the valuation results developed below apply to any pure jump innovation process, our primary interest is in innovation Y generated from the time-changed fBm X . For example, Y may be constructed by enriching the jump structure of X through an arbitrary predictable intensity function $\eta : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$, yielding

$$Y(t) = \int_0^t \int_{\mathbb{R}} \eta(s, x) \mu_X(ds, dx),$$

where μ_X is the jump measure of the time-changed fBm X . Alternatively, following the VG construction of [40], one may take the innovation process to be

$$Y(t) = W(t) = \theta \gamma(t) + \sigma B_H(\gamma(t)),$$

introduced in Section 4.4. Since W is likewise a pure jump process, the valuation framework developed here applies directly to this specification.

Taking logarithms of Equation (19), the log price process is linear in the innovation Y ,

$$\ln S(t) = \ln S(0) + \Xi(t) + Y(t). \quad (20)$$

A lower value of the Hurst exponent H produces rougher stock price trajectories, while a higher variance intensity v makes the stock price process less regular, exhibiting more pronounced periods of inactivity interspersed with abrupt jumps.

Under the statistical probability measure $\mathbb{P}$, the conditional expected log return of the stock over the interval $[u, t]$, given the information set $\mathcal{F}(u)$, is

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\ln \frac{S(t)}{S(u)} \middle| \mathcal{F}(u-) \right] &= \left(\mathbb{E}^{\mathbb{P}} [\Xi(t) | \mathcal{F}(u-)] - \Xi(u) \right) \\ &\quad + \left(\mathbb{E}^{\mathbb{P}} [A_Y(t) | \mathcal{F}(u-)] - A_Y(u) \right), \end{aligned} \quad (21)$$

where the compensator process A_Y depends on the latent activity time γ . The first term represents the contribution of the exogenous drift process Ξ , whereas the second captures the endogenous drift induced by the dependence structure of the innovation process Y .

More interestingly, when $H \neq 0.5$, log stock returns exhibit serial dependence arising from the endogenous drift component, characterized by a power-law decay of autocorrelations. For $H > 0.5$, log returns exhibit positive autocorrelation and long-range dependence, implying that periods of positive historical returns tend to be followed by higher expected future returns. Moreover, the influence of past increments decays slowly and remains significant even over long horizons. Conversely, when $H < 0.5$, increments are negatively correlated, giving rise to anti-persistent dynamics. In this regime, positive returns are more likely to be followed by negative returns and vice versa.

The conditional variance of the stock's log returns is given by

$$\text{Var}^{\mathbb{P}} \left[\ln \frac{S(t)}{S(u)} \middle| \mathcal{F}(u-) \right] = \text{Var}^{\mathbb{P}} \left[\Xi(t) - \Xi(u) + Y(t) - Y(u) \middle| \mathcal{F}(u-) \right]. \quad (22)$$

In many applications, the innovation process Y is assumed to be the sole source of randomness, with Ξ taken to be deterministic. Under this assumption, the conditional variance simplifies to the expected quadratic variation of the jump process Y ,

$$\begin{aligned} \text{Var}^{\mathbb{P}} \left[\ln \frac{S(t)}{S(u)} \middle| \mathcal{F}(u-) \right] &= \text{Var}^{\mathbb{P}} [Y(t) - Y(u) | \mathcal{F}(u-)] \\ &= \mathbb{E}^{\mathbb{P}} \left[\int_u^t \int_{\mathbb{R}} x^2 \nu_Y(ds, dx) \middle| \mathcal{F}(u) \right]. \end{aligned}$$

Importantly, the Hurst exponent H governs the temporal scaling of the variance. When $H > 0.5$, log returns are superdiffusive, with the variance growing faster than linearly over time. Conversely, when $H < 0.5$, log returns are subdiffusive, with the variance increasing at a rate slower than linear in time.

Remark 5.1 (Continuous Brownian motion). The model can be easily extended to include a continuous Brownian component. In this setting, the Itô formula for general semimartingales naturally accommodates the presence of a continuous

martingale term alongside the jump component. Nevertheless, such an extension is not pursued in the present study. Since the time-changed fBm generates an infinite-activity process, its sample paths are, in practice, virtually indistinguishable from those of a continuous process when observed at discrete time intervals. This observation is analogous to the argument contained in [40] in the context of the VG model. To maintain a concise exposition and focus on the principal contributions of this paper, we omit the additional source of randomness arising from additional Brownian motions.

5.2. No arbitrage and existence of an EMM. Let $\bar{S} = (\bar{S}(t))_{t \in [0, T]} = (e^{-\int_0^t r(c) dc} S(t))_{t \in [0, T]}$ denote the discounted stock price process, and let $\mathbb{Q}$ denote the equivalent martingale measure (EMM), defined as the probability measure $\mathbb{Q} \sim \mathbb{P}$ under which $\bar{S}$ is a $\mathbb{Q}$ -martingale under the filtration $\mathcal{F}$. The absence of arbitrage imposes a restriction on the drift of the stock price process under the statistical measure $\mathbb{P}$, as characterized in Proposition 5.2 below.

Proposition 5.2 (Stock price dynamics under EMM). *Suppose there exists an EMM $\mathbb{Q}$ whose compensator satisfies $\tilde{\nu}_Y(dt, dx) = \lambda(t, x) \nu_Y(dt, dx)$ for all $t \in [0, T]$, $x \in \mathbb{R}$, and for some predictable process $\lambda = (\lambda(t, x))_{t \in [0, T], x \in \mathbb{R}}$. Then, the stock's drift must satisfy the arbitrage-free drift condition*

$$r(t) = \xi(t) + \int_{\mathbb{R}} (e^x - 1) \lambda(t, x) \psi_Y(t, x) dx, \quad \forall t \in [0, T], \quad (23)$$

where λ denotes the market price of jump risk. Furthermore, under $\mathbb{Q}$, the discounted stock price $\bar{S}$ satisfies

$$\bar{S}(t) = \bar{S}(u) + \int_u^t \int_{\mathbb{R}} \bar{S}(c-) (e^x - 1) (\mu_Y - \tilde{\nu}_Y)(dc, dx), \quad 0 \leq u \leq t \leq T, \quad (24)$$

where $\tilde{\nu}_Y(dt, dx) = \lambda(t, x) \nu_Y(dt, dx)$.

From Equation (24), the conditional variance of the discounted stock price process is

$$\text{Var}^{\mathbb{Q}} [\bar{S}(t) | \mathcal{F}(u)] = \mathbb{E}^{\mathbb{Q}} \left[\int_u^t \int_{\mathbb{R}} \bar{S}(c-)^2 (e^x - 1)^2 \tilde{\nu}_Y(dc, dx) \middle| \mathcal{F}(u) \right]. \quad (25)$$

By substituting the arbitrage-free drift condition in Equation (23) into the stock price dynamics, the drift process Ξ can be endogenized, and the stock's log returns can be expressed as

$$\begin{aligned} \ln \frac{S(t)}{S(u)} &= \underbrace{\int_u^t r(c) dc}_{\text{Cumulative risk-free rate}} - \underbrace{\int_u^t \int_{\mathbb{R}} (e^x - 1) \lambda(c, x) \psi_Y(c, x) dx dc}_{\text{Risk premium}} + \underbrace{A_Y(t) - A_Y(u)}_{\text{Endogenous drift}} \\ &\quad + \underbrace{Y(t) - Y(u) - (A_Y(t) - A_Y(u))}_{\text{Martingale}}. \end{aligned} \quad (26)$$

The log return process therefore admits a decomposition into four distinct components: (i) the cumulative risk-free return, (ii) a risk premium determined by the market price of jump risk λ , (iii) an endogenous drift arising from the dependence structure and historical trajectory of the innovation process Y , and (iv) a martingale component capturing the residual uncertainty in stock returns.

While Proposition 5.2 yields the necessary condition on the drift of the stock when an EMM exists, the following result provides a partial converse to Proposition 5.2 by establishing sufficient conditions for the existence of an equivalent martingale measure.

Proposition 5.3 (Sufficient conditions for the existence of EMM). *Suppose there exists a predictable market price of jump risk process $\lambda = (\lambda(t, x))_{t \in [0, T], x \in \mathbb{R}}$ satisfying*

1. *Strict positivity:* $\lambda(t, x) > 0$ for all $t \in [0, T]$ and $x \in \mathbb{R}$,
2. *Novikov's condition:* $\mathbb{E}^{\mathbb{P}} \left[\exp \left(\frac{1}{2} \int_0^T \int_{\mathbb{R}} (\lambda(c, x) - 1)^2 \nu_Y(dc, dx) \right) \right] < \infty$,
3. *True martingale condition:* $\mathbb{E}^{\mathbb{P}} [\mathcal{L}_T \exp \{ \Xi(T) + Y(T) \}] = 1$, and
4. *Arbitrage-free drift condition:* $r(t) = \xi(t) + \int_{\mathbb{R}} (e^x - 1) \lambda(t, x) \psi_Y(t, x) dx$ for all $t \in [0, T]$.

Then, there exists an EMM $\mathbb{Q}$.

However, the presence of jumps in the stock price dynamics generally leads to market incompleteness. An implication of this proposition is that the EMM is generally non-unique, with the set of admissible EMMs consisting of equivalent probability measures $\mathbb{Q}(\lambda)$ corresponding to market prices of jump risk λ that satisfy the four stated hypotheses.

5.3. Derivative valuation. For valuation purposes, we work within an arbitrage-free market. By arbitrage-free, we mean that the market satisfies no free lunch with vanishing risk (NFLVR) and no dominance (ND). Under the fundamental theorems of asset pricing, these conditions imply the existence of an equivalent martingale measure (EMM) $\mathbb{Q}$. Specifically, we assume that the stock price process under the fVG model satisfies the sufficient conditions for the existence of an EMM established in Proposition 5.3.

However, the presence of jumps generally leads to market incompleteness. By the second fundamental theorem of asset pricing, the non-uniqueness of the EMM implies that the fVG market is incomplete. Therefore, derivative valuation requires the selection of an EMM $\mathbb{Q}(\lambda)$ from the admissible set. Several approaches have been proposed in the literature for selecting an EMM. One approach determines a unique derivative price through an optimization criterion, such as utility indifference pricing. A second approach selects a unique EMM based on economic considerations, such as the treatment of diversifiable risk, or analytical criteria, such as the minimal martingale measure. A third approach empirically identifies an EMM consistent with observed prices of traded derivatives.

Once an EMM $\mathbb{Q}$ has been selected through any of these approaches or otherwise, the arbitrage-free price at time t of a derivative with payoff function $G : \mathbb{R} \rightarrow \mathbb{R}$ and maturity T is given by the risk-neutral valuation formula

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(u) du} G(S(T)) \mid \mathcal{F}(t) \right].$$

This expectation can be evaluated numerically using the conditional distribution simulation methodology developed in Section 4.3.

5.4. Alternative time-changed fgBm model. In the preceding section, stock prices were specified as an exponential transformation of the innovation process Y ,

yielding a log return process that is linear in Y . Specifically, the fVG model was constructed by specifying

$$d \log S(t) = \xi(t) dt + dY(t).$$

Motivated by the classical Black–Scholes–Merton paradigm, an alternative and equally natural specification is to generalize geometric Brownian motion by replacing standard Brownian motion with the time-changed fBm process. This section considers this alternative formulation and demonstrates that the resulting model can be represented within the same valuation framework through an appropriate transformation of the jump intensity.

The time-changed fractional geometric Brownian motion (tfgBm) model is defined by

$$dS(t) = S(t)\xi(t) dt + S(t-) dY(t),$$

where $S(0) > 0$ is given. To ensure the positivity of the stock price process, we impose the condition that $\Delta Y(t) > -1$ for all $t \in [0, T]$. The corresponding solution S is given by the Doléans–Dade exponential,

$$\begin{aligned} S(t) &= S(0) \exp \{ \Xi(t) \} \prod_{u \in [0, t]} (1 + \Delta Y(u)) \\ &= S(0) \exp \left\{ \Xi(t) + \int_0^t \int_{\mathbb{R}} \ln(1+x) \mu_Y(du, dx) \right\}. \end{aligned} \quad (27)$$

Accordingly, the log return process admits the representation

$$\begin{aligned} \ln \frac{S(t)}{S(u)} &= \int_u^t \xi(c) dc + \int_u^t \int_{\mathbb{R}} \ln(1+x) \nu_Y(dc, dx) \\ &\quad + \int_u^t \int_{\mathbb{R}} \ln(1+x) (\mu_Y - \nu_Y)(dc, dx). \end{aligned}$$

The first term represents the exogenous drift, while the second term is the endogenous predictable drift induced by the innovation process Y . The final term is a $\mathbb{P}$ -martingale that captures the residual randomness in the log return dynamics.

Because the tfgBm framework is formulated directly through the stock price SDE, the derivation of the no-arbitrage drift condition is comparatively straightforward. Suppose there exists an EMM $\mathbb{Q}$. Then, under $\mathbb{Q}$, the compensated measure $\mu_Y - \tilde{\nu}_Y$ is a $\mathbb{Q}$ -martingale with $\tilde{\nu}_Y(dt, dx) = \lambda(t, x) \nu_Y(dt, dx)$, and the discounted stock price process $\bar{S}$ satisfies

$$\begin{aligned} \bar{S}(t) &= \bar{S}(u) + \int_u^t \bar{S}(c) \left[\xi(c) - r(c) + \int_{\mathbb{R}} x \lambda(c, x) \psi_Y(c, x) dx \right] dc \\ &\quad + \int_u^t \int_{\mathbb{R}} x \bar{S}(c-) (\mu_Y - \tilde{\nu}_Y)(dc, dx). \end{aligned}$$

Thus, the arbitrage-free drift condition is

$$r(t) = \xi(t) + \int_{\mathbb{R}} x \lambda(t, x) \psi_Y(t, x) dx, \quad \forall t \in [0, T]. \quad (28)$$

Similar to the fVG framework, the no-arbitrage condition yields an economically meaningful decomposition of the stock's log returns. In particular, $\ln(S(t)/S(u))$ can be expressed as the sum of four components: (i) the cumulative risk-free return

$\int_u^t r(c) \, dc$, (ii) the compensation for jump risk $\int_u^t \int_{\mathbb{R}} x \lambda(c, x) \psi_Y(c, x) \, dx \, dc$, (iii) an endogenous drift arising from past realizations $\int_u^t \int_{\mathbb{R}} \ln(1+x) \nu_Y(dc, dx)$, and (iv) residual martingale innovation process $\int_u^t \int_{\mathbb{R}} \ln(1+x) (\mu_Y - \nu_Y)(dc, dx)$. This decomposition is the direct analogue of Equation (26), with the principal distinction being the jump-amplitude transformation $x \mapsto \ln(1+x)$.

The price of a contingent claim with terminal payoff $G(S(T))$ remains $E^{\mathbb{Q}}[e^{-\int_t^T r(v) \, dv} G(S(T)) | \mathcal{F}(t)]$, where the terminal stock price under $\mathbb{Q}$ admits the representation

$$\begin{aligned} \ln \frac{S(t)}{S(u)} &= \int_u^t r(c) \, dc - \int_u^t \int_{\mathbb{R}} (x - \ln(1+x)) \lambda(c, x) \psi_Y(c, x) \, dx \, dc \\ &\quad + \int_u^t \int_{\mathbb{R}} \ln(1+x) (\mu_Y - \tilde{\nu}_Y)(dc, dx). \end{aligned}$$

and $\lambda(t, x)$ is determined by the no-arbitrage restriction in Equation (28).

A direct comparison of Equations (19) and (27) reveals that the fVG and tfgBm specifications are equivalent up to a deterministic transformation of the jump amplitudes. Let

$$Y_2(t) = \int_0^t \int_{\mathbb{R}} x \mu_{Y_2}(ds, dx)$$

denote the innovation process in the tfgBm model. Then, the same stock price dynamics can be obtained under the fVG framework by adopting the innovation process

$$Y_1(t) = \int_0^t \int_{\mathbb{R}} \ln(1+x) \mu_{Y_2}(ds, dx).$$

That is, the fVG innovation process Y_1 is obtained by applying the jump-amplitude transformation

$$\eta(x) := \ln(1+x)$$

to the jumps of the tfgBm innovation process Y_2 . Conversely, given an fVG innovation process $Y_1(t) = \int_0^t \int_{\mathbb{R}} x \mu_{Y_1}(ds, dx)$, the corresponding tfgBm innovation process is obtained by applying the inverse jump-amplitude transformation

$$\eta^{-1}(x) = e^x - 1.$$

Consequently, the principal results developed in Sections 5.1 through 5.3 extend directly to the tfgBm framework with the appropriate modification in the jump amplitude of the innovation process. Of the two specifications, the fVG model offers greater analytical tractability. Since log returns are linear in the driving innovation process Y , their statistical properties can be characterized more directly, facilitating both theoretical analysis and economic interpretation. By contrast, under the tfgBm specification, log returns depend on the modified jump amplitudes through the mapping $x \mapsto \ln(1+x)$, which complicates the characterization of return dynamics and the derivation of explicit pricing results.

6. Parameter estimation via GMM. This section presents a Generalized Method of Moments (GMM) procedure for estimating the parameters of the fVG model. We consider the constant-parameter stock price process

$$S(t) = S(0) \exp \{ \xi t + \theta \gamma(t; v) + \sigma B_H(\gamma(t; v)) \},$$

where $\xi \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\sigma \geq 0$, $v > 0$, and $H \in (0, 1)$; see Section 4.4. Several well-known models arise as special cases of the proposed evolution. When $H = 0.5$, the innovation process reduces to standard Brownian motion, and the model coincides with the classical VG model of [40]. Setting $v = 0$ and $\theta = 0$ eliminates the stochastic time-change, yielding the fractional Black–Scholes–Merton (BSM) model driven by fBm. Although this model is known to admit arbitrage opportunities, it is included for completeness and as a benchmark for the full specification. Finally, when $H = 0.5$, $\theta = 0$, and $v = 0$, the model further simplifies to the classical BSM framework.

Suppose stock prices are observed discretely as $\{S_{t_i}\}_{i=0}^N$ on the time grid $\{t_i\}_{i=0}^N$. For simplicity, we assume equally spaced observations with sampling interval $d > 0$, so that $t_i = id$ for all $i \in 0, 1, \dots, N$. Let $R_i^{(n)}$ denote the i -th observation of the n -lag log returns,

$$R_i^{(n)} := \ln(S_{t_{i+n}}/S_{t_i}),$$

where $n \in \mathbb{N}$, so that each lag corresponds to the time interval nd . Following Corollary 4.6, let $\rho(q, n)$ denote the unconditional mean (for $q = 1$) and central moments (for $q > 1$) of the n -lag log returns. Specifically,

$$\begin{aligned} \rho(q, n) &= \begin{cases} \mathbb{E} [R_i^{(n)}], & q = 1 \\ \mathbb{E} \left[\left(R_i^{(n)} - \mathbb{E} [R_i^{(n)}] \right)^q \right], & q \in \{2, 3, 4\} \end{cases} \\ &= \begin{cases} (\xi + \theta)nd, & q = 1 \\ \theta^2 v nd + \sigma^2 v^{2H} \frac{\Gamma(\frac{nd}{v} + 2H)}{\Gamma(\frac{nd}{v})}, & q = 2 \\ 2\theta^3 v^2 nd + 6H\sigma^2 \theta v^{1+2H} \frac{\Gamma(\frac{nd}{v} + 2H)}{\Gamma(\frac{nd}{v})}, & q = 3 \\ \theta^4 (3(nd)^2 v^2 + 6ndv^3) + 3\sigma^4 v^{4H} \frac{\Gamma(\frac{nd}{v} + 4H)}{\Gamma(\frac{nd}{v})} \\ \quad + 6\sigma^2 \theta^2 v^{2H+1} (nd + 2H(2H+1)v) \frac{\Gamma(\frac{nd}{v} + 2H)}{\Gamma(\frac{nd}{v})}, & q = 4. \end{cases} \end{aligned}$$

The corresponding sample moments are defined by

$$\hat{\rho}(q, n) = \begin{cases} \frac{1}{N-n+1} \sum_{i=0}^{N-n} R_i^{(n)}, & q = 1 \\ \frac{1}{N-n+1} \sum_{i=0}^{N-n} \left(R_i^{(n)} - \frac{1}{N-n+1} \sum_{j=0}^{N-n} R_j^{(n)} \right)^q, & q \in \{2, 3, 4\}. \end{cases} \quad (29)$$

Let $n_q \in \mathbb{N}$ denote the number of moment conditions of order q , and define $L = n_1 + n_2 + n_3 + n_4$. We then construct the $(L \times 1)$ vector of moment conditions as the stacked differences between theoretical and empirical moments,

$$\begin{aligned} \mathbf{m} = & \begin{bmatrix} \rho(1, 1) - \hat{\rho}(1, 1), & \dots & \rho(1, n_1) - \hat{\rho}(1, n_1), \\ \rho(2, 1) - \hat{\rho}(2, 1), & \dots & \rho(2, n_2) - \hat{\rho}(2, n_2), \\ \rho(3, 1) - \hat{\rho}(3, 1), & \dots & \rho(3, n_3) - \hat{\rho}(3, n_3), \\ \rho(4, 1) - \hat{\rho}(4, 1), & \dots & \rho(4, n_4) - \hat{\rho}(4, n_4) \end{bmatrix}^T. \end{aligned}$$

For a given $(L \times L)$ weighting matrix $\mathbf{W}$, the GMM estimator is defined as

$$\{\xi^*, \theta^*, v^*, H^*, \sigma^*\} = \underset{\xi, \theta, v, H, \sigma}{\operatorname{argmin}} \mathbf{m}^T \mathbf{W} \mathbf{m}. \quad (30)$$

The procedure is initialized with the identity weighting matrix, $\mathbf{W} = \mathbf{I}$. The GMM estimator is therefore obtained from

$$\{\xi_1^*, \theta_1^*, v_1^*, H_1^*, \sigma_1^*\} = \underset{\xi, \theta, v, H, \sigma}{\operatorname{argmin}} \sum_{q=1}^4 \sum_{n=1}^{n_q} (\rho(q, n) - \hat{\rho}(q, n))^2,$$

which corresponds to an equally weighted least-squares fit of the moment conditions.

To mitigate the risk of convergence to local minima, we adopt a multi-start optimization strategy at each GMM stage. Initial parameter values are selected from all combinations of $\xi_0 \in \{-1, 0, 1\}$, $\theta_0 \in \{-1, 0, 1\}$, $v_0 \in \{10^{-3}, 10^{-2}, 10^{-1}\}$, $H_0 \in \{0.3, 0.5, 0.7\}$, and $\sigma_0 \in \{0.05, 0.1, 0.2\}$. For each initialization, the optimization problem is solved independently, and the parameter estimates corresponding to the smallest value of the GMM objective function are retained.

7. An illustration using the S&P 500 index. To illustrate the fVG model and the proposed GMM estimation procedure, we use daily observations of the Standard & Poor's 500 Index (S&P 500) obtained from the CRSP database over a ten-year period spanning January 1, 2010 to December 31, 2019. For simplicity, observations are assumed to be equally spaced with sampling interval $d = 1/252$, corresponding to one trading day. The resulting dataset contains $N = 2516$ daily price observations.

For the empirical analysis, we set the moment truncation to $n_1 = n_2 = n_3 = n_4 = p$ with $p \in \{2, 3, 4, 5\}$, and implement the GMM procedure described in Section 6. For any given value of p , the GMM framework incorporates $4p$ moment conditions. Increasing p enables the moment conditions to capture the power-law temporal scaling more effectively, thereby improving identification and calibration of the Hurst exponent H . However, larger values of p can also introduce numerical challenges, including multicollinearity among the moments, increased sampling variability, and a more ill-conditioned optimization problem, which can adversely affect estimation stability. Consequently, for empirical applications, we recommend using $p = 2$, as it is the smallest value that avoids underidentification, providing eight moment conditions for the estimation of five parameters.

In addition to estimating the full model, we consider several restricted specifications obtained by imposing the constraints $\theta = 0$, $v = 0$, and $H = 0.5$ in all admissible combinations. For each restricted model, parameter estimation is carried out using the same GMM framework, with optimization performed over the corresponding constrained parameter space. We exclude the specification with $v = 0$ and $\theta \neq 0$. When $v = 0$, the parameter θ is no longer identified because it enters the model only through the first moment in conjunction with ξ . Consequently, only the sum $\xi + \theta$ is identifiable. Intuitively, this specification introduces an additional deterministic drift term, θt , into the log-return process, which is already captured by the existing deterministic drift ξt .

Although more efficient estimation procedures are available for certain nested specifications, particularly when $H = 0.5$, we adopt a common estimation methodology across all models to facilitate a direct comparison of parameter estimates. For example, the VG model admits likelihood-based estimation through its closed-form

PDF. Nevertheless, employing a unified GMM framework ensures that differences in empirical performance can be attributed to the model specifications themselves rather than to differences in the estimation methodology.

p = 2					p = 3				
ξ^*	σ^*	θ^*	v^*	H^*	ξ^*	σ^*	θ^*	v^*	H^*
0.1048	0.1451	-	-	-	0.1048	0.1443	-	-	-
0.1048	0.1451	-	0.0100	-	0.1048	0.1443	-	0.0100	-
0.5588	0.1411	-0.4540	0.0057	-	0.5533	0.1385	-0.4485	0.0080	-
0.1048	0.1225	-	-	0.4660	0.1048	0.1301	-	-	0.4776
0.1048	0.0768	-	0.0588	0.3197	0.1048	0.1307	-	0.0003	0.4786
0.5579	0.1256	-0.4531	0.0013	0.4716	0.5568	0.1233	-0.4520	0.0068	0.4717
p = 4					p = 5				
ξ^*	σ^*	θ^*	v^*	H^*	ξ^*	σ^*	θ^*	v^*	H^*
0.1048	0.1431	-	-	-	0.1048	0.1420	-	-	-
0.1048	0.1431	-	0.0100	-	0.1048	0.1420	-	0.0100	-
0.5531	0.1363	-0.4483	0.0094	-	0.5533	0.1337	-0.4485	0.0113	-
0.1048	0.1266	-	-	0.4720	0.1048	0.1242	-	-	0.4679
0.1048	0.0940	-	0.0855	0.3610	0.1048	0.0955	-	0.0691	0.3718
0.5358	0.1125	-0.4310	0.0104	0.4527	0.3397	0.1054	-0.2349	0.0236	0.4271

TABLE 1. Estimated GMM parameters for the S&P 500 under alternative model specifications and moment truncation levels $n_1 = n_2 = n_3 = n_4 = p$.

Table 1 reports the optimal GMM estimates for all six combinations of parameter restrictions across moment truncation levels $p \in \{2, 3, 4, 5\}$. Restricted parameters are indicated by a dash in the table.

We begin by discussing the main findings under our preferred specification, $p = 2$. When the Hurst exponent is fixed at $H = 0.5$, the estimated volatility parameter σ^* is approximately 0.14 across all model specifications. However, once H is estimated freely, the fitted value of σ^* declines. This pattern suggests that persistence captured by the Hurst exponent may otherwise be absorbed by the volatility parameter when temporal dependence is not adequately modeled. Stated differently, allowing H to vary attributes part of the observed variability in S&P 500 returns to long-range dependence rather than its instantaneous volatility.

When θ is fixed at zero, the estimated drift parameter ξ^* is consistently around 0.1. Once θ is allowed to vary, however, the optimal value of ξ^* increases because the first moment condition effectively identifies the combination $\xi^* + \theta^*$, which remains close to 0.1 across specifications. The estimated θ^* is generally negative, indicating negative return skewness. This finding is consistent with the negative sample skewness of the S&P 500 reported by [40] for the VG model.

The estimated variance-intensity parameter v^* is close to 0.01 when $\theta = 0$ and $H = 0.5$. Its value tends to increase when H is estimated freely and decrease when θ is unrestricted. Since v primarily governs the kurtosis of returns, this pattern

suggests that erroneous omission of serial persistence H leads to an underestimation of return kurtosis, whereas neglecting the skewness parameter θ causes excess kurtosis to absorb part of the asymmetry in the return distribution.

When H is unrestricted, the estimated Hurst exponent generally lies within the interval $[0.3, 0.5]$. Although these values remain relatively close to the Brownian benchmark 0.5, the deviations are significant and economically meaningful. As noted above, even modest departures from $H \neq 0.5$ materially affect the estimation of the remaining parameters and imply substantially different stock-price dynamics.

Under the fractional BSM specification ($\theta = v = 0$), the estimated Hurst exponent is typically around 0.47. Because higher-order cumulants vanish when $\theta = v = 0$, estimation relies primarily on the first two moments, with H entering through the second central moment. The results therefore suggest a slightly slower-than-linear growth in variance scaling over longer horizons relative to the Brownian benchmark. Moreover, when kurtosis is explicitly modeled through the parameter v , the estimated Hurst exponent tends to decline further below 0.5 to 0.32. This finding indicates that part of the non-Gaussian heavy-tailed behavior of returns may otherwise be misattributed to temporal dependence, as both kurtosis and the Hurst exponent influence higher-order return moments. In contrast, under the fully unrestricted specification, the estimated Hurst exponent increases to approximately 0.47 once skewness and kurtosis are jointly accommodated through θ and v . Our result shows that the accurate calibration of H requires that skewness and kurtosis are appropriately modeled.

Across different values of p , the qualitative findings remain largely unchanged. In particular, the parameter estimates, especially for ξ^* , σ^* and H^* , are relatively stable within a given model specification. Nevertheless, estimation becomes less stable as p increases beyond 5. For example, when $p = 10$ (not shown), we occasionally obtain positive estimates of θ^* despite the negative empirical skewness of S&P 500 returns. Parameter estimates often hit the boundaries and fluctuate erratically across initial starting values. Such results are consistent with the numerical difficulties discussed earlier and reinforce our preference for moderate truncation levels, particularly $p \in \{2, 3\}$.

Interestingly, the estimated Hurst exponent is consistently below the Brownian benchmark of $H = 0.5$. This finding suggests mild anti-persistence in S&P 500 returns, whereby positive shocks are more likely to be followed by offsetting movements. Moreover, because the model separately accounts for skewness and excess kurtosis, the estimated Hurst exponent is less likely to absorb non-Gaussian features of the return distribution. Consequently, the results indicate that apparent long memory documented in simpler specifications may partly reflect unmodeled heavy tails rather than genuine temporal persistence. Nonetheless, although $H < 0.5$ is often interpreted as evidence of anti-persistent increments, the present GMM procedure identifies H primarily through the scaling of return moments across investment horizons. Consequently, the estimates are more naturally interpreted as indicating that return moments accumulate slightly more slowly than the linear growth implied by standard Brownian motion. In other words, the term structure of empirical moments exhibits mildly sublinear scaling, which the model captures through a Hurst exponent below 0.5.

A formal comparison of model performance remains challenging. The conditional distribution of the time-changed fractional Brownian motion does not admit

a tractable closed-form expression, making the computation of forecasting metrics, likelihood-based criteria, and option-pricing implications considerably more demanding. A comprehensive evaluation of predictive and pricing performance is therefore left for future research.

8. Conclusion. This paper introduces a novel framework for equity option pricing in which the stock prices are driven by a time-changed fBm. The proposed process preserves many of the key statistical properties that make fBm attractive for financial modeling, including its ability to capture long-range dependence, rough-path-type behavior, and anomalous diffusion. At the same time, the introduction of a stochastic time change ensures that time-changed fBm is a semimartingale process, thereby permitting arbitrage-free valuation within the standard framework of mathematical finance.

The proposed model provides a parsimonious yet rich framework for capturing salient features of observed asset-price dynamics. We hope that the results presented in this paper will stimulate further empirical and theoretical investigation, especially into the applications of asset pricing, risk management, and derivative valuation.

REFERENCES

- [1] P. Abry and D. Veitch, [Wavelet analysis of long-range-dependent traffic](#), *IEEE Transactions on Information Theory*, **44** (1998), 2-15.
- [2] C. Bender, T. Sottinen and E. Valkeila, [Pricing by hedging and no-arbitrage beyond semimartingales](#), *Finance and Stochastics*, **12** (2008), 441-468.
- [3] T. Björk and H. Hult, A note on wick products and the fractional black-scholes model, *Finance and Stochastics*, **9** (2005), 197-209.
- [4] L. Calvet and A. Fisher, [Multifractality in asset returns: Theory and evidence](#), *The Review of Economics and Statistics*, **84** (2002), 381-406.
- [5] P. Caraianni, [Evidence of multifractality from emerging european stock markets](#), *PLOS ONE*, **7** (2012), e40693.
- [6] A. Charles and O. Darné, [Variance-ratio tests of random walk: An overview](#), *Journal of Economic Surveys*, **23** (2009), 503-527.
- [7] P. Cheridito, [Mixed fractional Brownian motion](#), *Bernoulli*, **7** (2001), 913-934.
- [8] P. Cheridito, [Arbitrage in fractional Brownian motion models](#), *Finance and Stochastics*, **7** (2003), 533-553.
- [9] C. Christodoulou-Volos and F. M. Siokis, Long range dependence in stock market returns, *Applied Financial Economics*, **16** (2006), 1331-1338.
- [10] R. Cont and P. Tankov, [Financial Modelling with Jump Processes](#), Chapman and Hall/CRC, (December 30, 2003).
- [11] D. J. Daley and D. Vere-Jones, [An Introduction to the Theory of Point Processes](#) (J. Gani, C. C. Heyde, P. Jagers, & T. G. Kurtz). Springer New York, 2008.
- [12] L. Decreusefond and A. S. Üstünel, [Fractional Brownian motion: Theory and applications](#), *ESAIM: Proceedings*, **5** (1998), 75-86.
- [13] F. Delbaen and W. Schachermayer, A General Version of the Fundamental Theorem of Asset Pricing. *Mathematische Annalen*, **300** (1994), 463-520.
- [14] T. Di Matteo, [Multi-scaling in finance](#), *Quantitative Finance*, **7** (2007), 21-36.
- [15] R. J. Elliott and J. Van Der Hoek, A general fractional white noise theory and applications to finance, *Mathematical Finance*, **13** (2003), 301-330.
- [16] R. Finlay and E. Seneta, Stationary-increment student and variance-gamma processes, *Journal of Applied Probability*, **43** (2006), 441-453. Retrieved September 19, 2026, from <https://www.jstor.org/stable/27595742>

- [17] R. Finlay and E. Seneta, A Gamma activity time process with noninteger parameter and self-similar limit, *Journal of Applied Probability*, **44** (2007), 950-959. Retrieved September 19, 2026, from <https://www.jstor.org/stable/27595899>
- [18] R. Finlay, E. Seneta and D. Wang, An inverse Gamma activity time process with noninteger parameters and a self-similar limit, *Journal of Applied Probability*, **49** (2012), 441-450.
- [19] C. Fox, Some applications of mellin transforms to the theory of bivariate statistical distributions, *Mathematical Proceedings of the Cambridge Philosophical Society*, **53** (1957), 620-628.
- [20] C. Franzke, Long-range dependence and climate noise characteristics of antarctic temperature data, *Journal of Climate*, **23** (2010), 6074-6081.
- [21] H. Geman, D. B. Madan and M. Yor, Time changes for Lévy processes, *Mathematical Finance*, **11** (2001), 79-96.
- [22] M. T. Greene and B. D. Fielitz, Long-term dependence in common stock returns, *Journal of Financial Economics*, **4** (1977), 339-349.
- [23] M. Grossglauser and J.-C. Bolot, On the relevance of long-range dependence in network traffic, *IEEE/ACM Transactions on Networking*, **7** (1999), 629-640.
- [24] P. Guasoni, No arbitrage under transaction costs, with fractional brownian motion and beyond, *Mathematical Finance*, **16** (2006), 569-582.
- [25] D. Heath, S. Resnick and G. Samorodnitsky, Heavy tails and long range dependence in on/off processes and associated fluid models, *Mathematics of Operations Research*, **23** (1998), 145-165.
- [26] C. C. Heyde and N. N. Leonenko, Student processes, *Advances in Applied Probability*, **37** (2005), 342-365.
- [27] H. A. A. B. Hoque, J. H. Kim and C. S. Pyun, A comparison of variance ratio tests of random walk: A case of asian emerging stock markets, *International Review of Economics & Finance*, **16** (2007), 488-502.
- [28] Y. Hu and B. Øksendal, Fractional white noise calculus and applications to finance, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, **06** (2003), 1-32.
- [29] H. E. Hurst, Long-term storage capacity of reservoirs, *Transactions of the American Society of Civil Engineers*, **116** (1951), 770-799.
- [30] T. Iliopoulou, S. M. Papalexou, Y. Markonis and D. Koutsoyiannis, Revisiting Long-Range Dependence in Annual Precipitation. *Journal of Hydrology*, **556** (2018), 891-900.
- [31] J. Jacod and A. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, (2003).
- [32] R. A. Jarrow, *Continuous-Time Asset Pricing Theory. A Martingale-Based Approach* (2nd ed.), Springer, (2021).
- [33] R. A. Jarrow, P. Protter and H. Sayit, No Arbitrage without Semimartingales, *The Annals of Applied Probability*, **19** (2009), 596-616.
- [34] D. Koutsoyiannis, Climate Change, the Hurst Phenomenon, and Hydrological Statistics, *Hydrological Sciences Journal*, **48** (2003), 3-24.
- [35] T. J. Kozubowski, M. M. Meerschaert and K. Podgorski, Fractional Laplace motion, *Advances in Applied Probability*, **38** (2006), 451-464.
- [36] A. Kumar, A. Wylomańska, R. Poloczański and S. Sundar, Fractional Brownian motion time-changed by Gamma and inverse Gamma process, *Physica A: Statistical Mechanics and Its Applications*, **468** (2017), 648-667.
- [37] W. Leland, M. Taqqu, W. Willinger and D. Wilson, On the self-similar nature of ethernet traffic, *IEEE/ACM Transactions on Networking*, **2** (1994), 1-15.
- [38] A. W. Lo and A. C. MacKinlay, Stock market prices do not follow random walks: Evidence from a simple specification test, *The Review of Financial Studies*, **1** (1988), 41-66. Retrieved July 21, 2026, from <https://www.jstor.org/stable/2962126>.
- [39] A. W. Lo and A. C. MacKinlay, The size and power of the variance ratio test in finite samples: A monte carlo investigation, *Journal of Econometrics*, **40** (1989), 203-238.
- [40] D. B. Madan, P. P. Carr and E. C. Chang, The variance Gamma process and option pricing, *Review of Finance*, **2** (1998), 79-105.

- [41] D. B. Madan and E. Seneta, The variance Gamma (V.G.) model for share market returns, *The Journal of Business*, **63** (1990), 511-524. Retrieved September 19, 2026, from <https://www.jstor.org/stable/2353303>
- [42] D. B. Madan and K. Wang, The economics of time as it is embedded in the prices of options §, *Quantitative Finance*, **23** (2023), 579-593.
- [43] B. Mandelbrot, A. Fisher and L. Calvet, A multifractal model of asset returns, *Cowles Foundation Discussion Papers*, (1997), <https://elischolar.library.yale.edu/cowles-discussion-paper-series/1412>.
- [44] B. B. Mandelbrot and J. W. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Review*, **10** (1968), 422-437. Retrieved September 19, 2026, from <https://www.jstor.org/stable/2027184>
- [45] B. B. Mandelbrot and J. R. Wallis, Noah, Joseph, and operational hydrology, *Water Resources Research*, **4** (1968), 909-918.
- [46] R. N. Mantegna and H. E. Stanley, Scaling behaviour in the dynamics of an economic index, *Nature*, **376** (1995), 46-49.
- [47] C. Park, F. Hernández-Campos, L. Le, J. S. Marron, J. Park, V. Pipiras, F. D. Smith, R. L. Smith, M. Trovero and Z. Zhu, Long-range dependence analysis of internet traffic, *Journal of Applied Statistics*, **38** (2011), 1407-1433.
- [48] J. D. Pelletier and D. L. Turcotte, Long-range persistence in climatological and hydrological time series: analysis, modeling and application to drought hazard assessment, *Journal of Hydrology*, **203** (1997), 198-208.
- [49] C.-K. Peng, S. V. Buldyrev, A. L. Goldberger, S. Havlin, F. Sciortino, M. Simons and H. E. Stanley, Long-range correlations in nucleotide sequences, *Nature*, **356** (1992), 168-170.
- [50] P. E. Protter, Semimartingales and stochastic integrals, In *Stochastic Integration and Differential Equations*, Springer, (2005), 51-100.
- [51] B. K. Ray and R. S. Tsay, Long-range dependence in daily stock volatilities, *Journal of Business & Economic Statistics*, **18** (2000), 254-262.
- [52] L. C. G. Rogers, Arbitrage with fractional brownian motion, *Mathematical Finance*, **7** (1997), 95-105.
- [53] H.-J. Ryoo and G. Smith, Korean stock prices under price limits: Variance ratio tests of random walks, *Applied Financial Economics*, **12** (2002), 545-553.
- [54] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, (Chapman & Hall, Red.). CRC Press LLC, 1994.
- [55] G. Smith, K. Jefferis and H.-J. Ryoo, African stock markets: Multiple variance ratio tests of random walks, *Applied Financial Economics*, **12** (2002), 475-484.
- [56] G. Smith and H.-J. Ryoo, Variance ratio tests of the random walk hypothesis for european emerging stock markets, *The European Journal of Finance*, **9** (2003), 290-300.
- [57] R. L. Smith, Long-range dependence and global warming, In V. Barnett and F. Turkman (Eds.) *Statistics for the Environment*, (1993), 141-161.
- [58] P. Suárez-García and D. Gómez-Ullate, Multifractality and long memory of a financial index, *Physica A: Statistical Mechanics and its Applications*, **394** (2014), 226-234.
- [59] E. Tagliazucchi, F. von Wegner, A. Morzelewski, V. Brodbeck, K. Jahnke and H. Laufs, Breakdown of long-range temporal dependence in default mode and attention networks during deep sleep, *Proceedings of the National Academy of Sciences*, **110** (2013), 15419-15424. <https://pmc.ncbi.nlm.nih.gov/articles/PMC3780893/pdf/pnas.201312848.pdf>
- [60] J. Tan, *Pricing Weather Contracts Under Persistent Memory in Temperature*, Journal of Risk and Insurance, (Forthcoming), 2026.
- [61] J. Tan and R. A. Jarrow, *Risk-Neutral Valuation with a Non-Traded Fractional Brownian Underlying*, SSRN 5692102, 2026.
- [62] A. Turiel and C. J. Pérez-Vicente, Multifractal geometry in stock market time series, *Physica A: Statistical Mechanics and its Applications*, **322** (2003), 629-649.

- [63] L. Wang, M. C. Chiu and H. Y. Wong, [Volterra mortality model: actuarial valuation and risk management with long-range dependence](#), *Insurance: Mathematics and Economics*, **96** (2021), 1-14.
- [64] X. Wang, W. Xiao and J. Yu, Modeling and forecasting realized volatility with the fractional ornstein–uhlenbeck process, *Journal of Econometrics*, **232** (2023), 389-415.
- [65] W. Willinger, M. S. Taqqu and V. Teverovsky, [Stock market prices and long-range dependence](#), *Finance and Stochastics*, **3** (1999), 1-13.
- [66] A. T. A. Wood and G. Chan, [Simulation of stationary gaussian processes in \$\[0, 1\]\$ d](#), *Journal of Computational and Graphical Statistics*, **3** (1994), 409-432.
- [67] E. M. Wright, [On the coefficients of power series having exponential singularities](#), *Journal of the London Mathematical Society*, **8** (1933), 71-79.
- [68] M. Yor, Some Remarkable Properties of Gamma Processes, In M. C. Fu, R. A. Jarrow, J.-Y. J. Yen, & R. J. Elliott (Eds.), *Advances in Mathematical Finance*, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Inc., Boston, MA, Birkhäuser, (2007), 37-47.
- [69] H. Zhou, K. Q. Zhou and X. Li, Stochastic mortality dynamics driven by mixed fractional Brownian motion, *Insurance: Mathematics and Economics*, **106** (2022), 218-238.

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Appendix A. Proofs.

A.1. **Proof of Proposition 4.1.** For any $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{E}[B_H(\gamma(t))^n] &= \mathbb{E}[\mathbb{E}[B_H(\gamma(t))^n | \gamma(t)]] = \begin{cases} \mathbb{E}[\gamma(t)^{nH}(n-1)!!] \\ 0 \end{cases} \\ &= \begin{cases} v^{nH} \frac{\Gamma(t/v + nH)}{\Gamma(t/v)} (n-1)!! , & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \end{aligned}$$

where the second equality follows from the central moments of the Gaussian distribution and the third equality follows from the raw moments of the Gamma process.

A.2. **Proof of Theorem 4.2.** Since the path of B_H is continuous and γ is a pure jump process, $X = B_H(\gamma)$ is also a pure jump process. In addition, X is adapted and cadlag because B_H and γ are adapted and cadlag. Furthermore, by Proposition 2 of [68], X has bounded variation on compacts. Thus, by Theorem 7 of [50] (Chapter II, p. 55), X is a semimartingale.

A.3. **Proof of Proposition 4.3.** By Proposition 2 of [68], X has bounded variation on compacts, and thus, the jumps of X are bounded over $[0, T]$. It follows from Theorem 35 of [50] (Chapter II, p.131) that X is a special semimartingale. Thus, by Theorem 34 of [50] (Chapter II, p.130), X admits a semimartingale decomposition $X = M + A$, where M is a local martingale and A is a unique, predictable, finite variation process.

Next, we construct the compensator process $A = (A(t))_{t \in [0, T]}$. By the marked point process representation of A , we have

$$A(t) = \int_0^t \int_{\mathbb{R}} x \nu_X(ds, dx)$$

$$\begin{aligned}
&= \int_0^t \int_{\mathbb{R}} x \psi_X(s, x) \, dx \, ds \\
&= \int_0^t \int_{\mathbb{R}} x \left(\int_0^\infty \phi(x; \gamma(s), g) \psi_\gamma(g) \, dg \right) \, dx \, ds \\
&= \int_0^t \int_0^\infty \left(\int_{\mathbb{R}} x \phi(x; \gamma(s), g) \, dx \right) \psi_\gamma(g) \, dg \, ds \\
&= \int_0^t \int_0^\infty m_B(g; \gamma(s)) \psi_\gamma(g) \, dg \, ds
\end{aligned}$$

for all $t \in [0, T]$. The existence of integrand also implies that A is absolutely continuous, and thus, continuous.

Lastly, we show that $X - A$ is a true martingale. A sufficient condition is to show that $E[(X(t) - A(t))^2] < \infty$ for all $t \in [0, T]$. By Minkowski's inequality

$$E[(X(t) - A(t))^2]^{\frac{1}{2}} \leq E[X(t)^2]^{\frac{1}{2}} + E[A(t)^2]^{\frac{1}{2}},$$

it suffices to bound each term, $E[X(t)^2]$ and $E[A(t)^2]$, uniformly across $t \in [0, T]$. Firstly, using the iterated law of expectation and the fractional moments of the gamma distribution,

$$E[X(t)^2] = E[E[B_H(\gamma(t))^2 | \gamma(t)]] = E[\gamma(t)^{2H}] = \frac{\Gamma(t/v + 2H)}{\Gamma(t/v)} v^{2H} < \infty$$

for all $t \in [0, T]$, $v > 0$ and $H \in (0, 1)$.

Next, to bound $E[A(t)^2]$, recall that

$$A(t) = \int_0^t a_X(s) \, ds \quad \text{with} \quad a_X(s) = \int_0^\infty m_B(g; \gamma(s)) \psi_\gamma(g) \, dg.$$

For any $g \geq 0$, applying Jensen's inequality for conditional expectations to $m_B(g; \gamma(s)) = E[B_H(\gamma(s) + g) - B_H(\gamma(s)) | \mathcal{F}_{\gamma(s)}^B]$ gives

$$m_B(g; \gamma(s))^2 \leq E[(B_H(\gamma(s) + g) - B_H(\gamma(s)))^2 | \mathcal{F}_{\gamma(s)}^B].$$

where $\mathcal{F}_{\gamma(s)}^B$ denotes the filtration generated by B_H up to time $\gamma(s)$. Taking unconditional expectations and using the stationary increment property of fBm yields

$$(\|m_B(g; \gamma(s))\|_{L^2})^2 \leq E[(B_H(\gamma(s) + g) - B_H(\gamma(s)))^2] = g^{2H}.$$

Then, applying Minkowski's integral inequality to $a_X(s)$ and integrating against the gamma compensator density $\psi_\gamma(g) = \frac{e^{-g/v}}{vg}$, we obtain

$$\|a_X(s)\|_{L^2} \leq \int_0^\infty \|m_B(g; \gamma(s))\|_{L^2} \psi_\gamma(g) \, dg \leq \int_0^\infty g^H \frac{e^{-g/v}}{vg} \, dg = v^{H-1} \Gamma(H).$$

Finally, Minkowski's inequality applied to $A(t) = \int_0^t a_X(s) \, ds$ yields $\|A(t)\|_{L^2} \leq tv^{H-1} \Gamma(H)$, so that

$$E[A(t)^2] \leq t^2 v^{2(H-1)} \Gamma(H)^2 < \infty, \quad t \in [0, T], v > 0, H \in (0, 1).$$

This concludes the proof for elevating the local martingale into a true martingale.

A.4. **Proof of Proposition 4.5.** For any $t \geq 0$ and $h \geq 0$, the increment of W is

$$W(t+h) - W(t) = \theta G_h + \sigma G_h^H Z,$$

where $G_h = \gamma(t+h) - \gamma(t) \sim \Gamma(h/v, 1/v)$ and $Z \sim N(0, 1)$ are independent. Then, the n -th raw moment of the increment $W(t+h) - W(t)$ is

$$\begin{aligned} & \mathbb{E}[(W(t+h) - W(t))^n] \\ &= \mathbb{E}[(\theta G_h + \sigma G_h^H Z)^n] \\ &= \mathbb{E}\left[\sum_{j=0}^n \binom{n}{j} (\theta G_h)^{n-j} (\sigma G_h^H Z)^j\right] \\ &= \sum_{j=0}^n \binom{n}{j} \theta^{n-j} \sigma^j \mathbb{E}[G_h^{n-j(1-H)}] \mathbb{E}[Z^j] \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \theta^{n-2k} \sigma^{2k} \mathbb{E}[G_h^{n-2k(1-H)}] \mathbb{E}[Z^{2k}] \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \theta^{n-2k} \sigma^{2k} v^{n-2k(1-H)} \frac{\Gamma(\frac{h}{v} + n - 2k(1-H))}{\Gamma(\frac{h}{v})} (2k-1)!! \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k! (n-2k)!} \theta^{n-2k} \sigma^{2k} v^{n-2k(1-H)} \frac{\Gamma(\frac{h}{v} + n - 2k(1-H))}{\Gamma(\frac{h}{v})}. \end{aligned}$$

Next, to establish the central moments, we evaluate the expectation

$$\mathbb{E}[W(t+h) - W(t)] = \theta \mathbb{E}[G_h] = \theta h$$

and express the centered increment as

$$W(t+h) - W(t) - \mathbb{E}[W(t+h) - W(t)] = \theta(G_h - h) + \sigma G_h^H Z.$$

Consequently, for any $n \in \{2, 3, \dots\}$, the n -th central moment is

$$\begin{aligned} & \mathbb{E}[(W(t+h) - W(t) - \mathbb{E}[W(t+h) - W(t)])^n] \\ &= \mathbb{E}[(\theta(G_h - h) + \sigma G_h^H Z)^n] \\ &= \mathbb{E}\left[\sum_{j=0}^n \binom{n}{j} (\theta(G_h - h))^{n-j} (\sigma G_h^H Z)^j\right] \\ &= \sum_{j=0}^n \binom{n}{j} \theta^{n-j} \sigma^j \mathbb{E}[(G_h - h)^{n-j} G_h^j] \mathbb{E}[Z^j] \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \theta^{n-2k} \sigma^{2k} \mathbb{E}[(G_h - h)^{n-2k} G_h^{2k}] \mathbb{E}[Z^{2k}] \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \theta^{n-2k} \sigma^{2k} \sum_{\ell=0}^{n-2k} \binom{n-2k}{\ell} (-h)^{n-2k-\ell} v^{2kH+\ell} \frac{\Gamma(\frac{h}{v} + 2kH + \ell)}{\Gamma(\frac{h}{v})} (2k-1)!! \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!} \theta^{n-2k} \sigma^{2k} \sum_{\ell=0}^{n-2k} \frac{1}{\ell! (n-2k-\ell)!} (-h)^{n-2k-\ell} v^{2kH+\ell} \frac{\Gamma(\frac{h}{v} + 2kH + \ell)}{\Gamma(\frac{h}{v})} \end{aligned}$$

since for any $k \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$,

$$\begin{aligned} \mathbb{E} [(G_h - h)^{n-2k} G_h^{2kH}] &= \mathbb{E} \left[G_h^{2kH} \sum_{\ell=0}^{n-2k} \binom{n-2k}{\ell} (-h)^{n-2k-\ell} G_h^\ell \right] \\ &= \sum_{\ell=0}^{n-2k} \binom{n-2k}{\ell} (-h)^{n-2k-\ell} \mathbb{E} [G_h^{2kH+\ell}] \\ &= \sum_{\ell=0}^{n-2k} \binom{n-2k}{\ell} (-h)^{n-2k-\ell} v^{2kH+\ell} \frac{\Gamma(\frac{h}{v} + 2kH + \ell)}{\Gamma(\frac{h}{v})}, \end{aligned}$$

and the coefficients simplify to

$$\binom{n}{2k} \binom{n-2k}{\ell} (2k-1)!! = \frac{n!}{2^k k! (n-2k)!} \frac{(n-2k)!}{\ell! (n-2k-\ell)!} = \frac{n!}{2^k k! \ell! (n-2k-\ell)!}.$$

A.5. Proof of Proposition 5.2. We first recall the generalized Ito formula for semimartingale by [50].

Lemma A.1 (Ito formula for semimartingale, Theorem 32 of [50]). *Let Y be a semimartingale and $f \in C^2(\mathbb{R})$. Then, the generalized Ito formula is*

$$\begin{aligned} f(Y_t) &= f(Y_0) + \int_0^t \frac{\partial f}{\partial y}(Y_{s-}) dY_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial y^2}(Y_{s-}) d[Y]_s^c \\ &\quad + \sum_{s \in (0, t]} \left[f(Y_s) - f(Y_{s-}) - \frac{\partial f}{\partial y}(Y_{s-}) \Delta Y_s \right]. \end{aligned}$$

Following the log returns process in Equation (20), the discounted log returns process is given by

$$d \ln \bar{S}(t) = (\xi(t) - r(t)) dt + dY(t), \quad \forall t \in [0, T].$$

Applying the generalized Ito formula in Lemma A.1 with $f(x) = e^x$ yields the discounted stock price process

$$\bar{S}(t) = \bar{S}(0) + \int_0^t \bar{S}(c) (\xi(c) - r(c)) dc + \int_0^t \int_{\mathbb{R}} \bar{S}(c-) (e^x - 1) \mu_Y(dc, dx).$$

Under $\mathbb{Q}$, the stock price process can be decomposed into its predictable drift and martingale components

$$\begin{aligned} \bar{S}(t) &= \bar{S}(0) + \int_0^t \bar{S}(c) (\xi(c) - r(c)) dc + \int_0^t \int_{\mathbb{R}} \bar{S}(c) (e^x - 1) \tilde{\nu}_Y(dc, dx) \\ &\quad + \int_0^t \int_{\mathbb{R}} \bar{S}(c-) (e^x - 1) (\mu_Y - \tilde{\nu}_Y)(dc, dx) \\ &= \bar{S}(0) + \int_0^t \bar{S}(c) \left(\xi(c) - r(c) + \int_{\mathbb{R}} (e^x - 1) \lambda(c, x) \psi_Y(c, x) dx \right) dc \\ &\quad + \int_0^t \int_{\mathbb{R}} \bar{S}(c-) (e^x - 1) (\mu_Y - \tilde{\nu}_Y)(dc, dx) \end{aligned}$$

where $\tilde{\nu}_Y(dt, dx) = \lambda(t, x)\nu_Y(dt, dx)$. Since $\bar{S}$ is a martingale under $\mathbb{Q}$, its predictable finite-variation component must vanish identically. Consequently,

$$r(t) = \xi(t) + \int_{\mathbb{R}} (e^x - 1) \lambda(t, x) \psi_Y(t, x) dx,$$

which is precisely the arbitrage-free drift condition. Substituting this identity into the above decomposition yields Equation (24).

A.6. Proof of Proposition 5.3. Under the strict positivity and Novikov conditions, Girsanov's theorem for marked point processes implies that the stochastic exponential

$$\frac{d\mathbb{Q}(\lambda)}{d\mathbb{P}} = \exp \left\{ - \int_0^T \int_{\mathbb{R}} (\lambda(s, x) - 1) \nu_Y(ds, dx) + \int_0^T \int_{\mathbb{R}} \log \lambda(s, x) \mu_Y(ds, dx) \right\}$$

is a true $\mathbb{P}$ -martingale. Consequently, it defines a probability measure $\mathbb{Q}(\lambda)$ equivalent to $\mathbb{P}$.

Furthermore, under $\mathbb{Q}(\lambda)$, the compensator of the jump measure μ_Y is given by

$$\tilde{\nu}_Y(dt, dx) = \lambda(t, x) \nu_Y(dt, dx).$$

Substituting this compensator into the stock price dynamics, the arbitrage-free drift condition ensures that the drift of the discounted stock price process vanishes. Finally, the true martingale condition guarantees that the discounted stock price is a true $\mathbb{Q}(\lambda)$ -martingale. Thus, $\mathbb{Q}(\lambda)$ is an EMM.