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The Valuation and Hedging of Non-Maturity Deposits: Frequently Asked Questions

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Abstract

A recent paper by Jarrow, Melnikov, Sherman, van Deventer and Zorn [2022] showed how the discrete time valuation formulas for non-maturity deposits, following Jarrow and van Deventer [1996], can be extended and generalized using risk-neutral valuation. These formulas apply for any number of interest rate risk factors and any form of interest rate volatility as long as the no arbitrage constraints of Heath, Jarrow and Morton[1992] apply. The resulting formulas often have closed form solutions. Even when those closed form solutions become more complex, one can easily calculate a numerical solution that requires only the existing output from a Heath, Jarrow and Morton simulation of risk-free yields plus regression equations that explain the response of deposit rates and balances to changes in the economic environment going forward.

The paper generated a number of thoughtful questions from practitioners and academics. In this note, we provide brief answers to six of the questions received.

The Valuation and Hedging of Non-Maturity Deposits:
Frequently Asked Questions

A recent paper by Jarrow, Melnikov, Sherman, van Deventer and Zorn [2022] on the valuation and hedging of non-maturity deposits has prompted a number of interesting questions from practitioners and risk management experts. In this note, we provide brief answers to six of the questions received. This note presumes that an analyst has access to a Monte Carlo simulation of risk-free yields from which the deposit valuations can be derived. The simulation should be done in a manner consistent with the no-arbitrage conditions of Heath, Jarrow and Morton [1992]. A recent simulation of the U.S. Treasury curve by van Deventer [November 10, 2022] is used in this note. The van Deventer simulation uses 91-day time steps and extends 30 years forward from November 10, 2022. Given the base U.S. Treasury simulation, changes in the analysis of non-maturity deposits does not require a new Treasury simulation. Non-maturity deposit analysis can be done simply by reading the output from the existing Treasury simulation and applying the parameters and assumptions for non-maturity deposit analysis to derive the valuation and hedge ratios for the non-maturity deposit franchise.

Section 1 of this note summarizes the six questions posed to the authors in response to the 2022 paper. Appendices A and B are repeated from the original paper to show the derivation of the non-maturity deposit model. Appendix C contains extensions of that model derived in a similar way.

We now turn to the question and answer session.

1. Six Frequently Asked Questions about Deposit Valuation and Hedging

1. How would the model be modified if the deposit rate and deposit balances were a function not just of the current short-term risk-free rate but also lagged values of the risk-free rate?

The use of lagged values of the risk-free rate and prior deposit rates and balances is common in the analysis of non-maturity deposits. The deposit model by Jarrow and van Deventer [1996] and its discrete exposition by Jarrow, Melnikov, Sherman, van Deventer and Zorn (“JMSVZ”) value each element of deposit cash flows and related marginal operating expenses by discounting using the risk-free money market fund from the risk-free rate simulation. The money fund $B(t,t+N\Delta)$ represents the compounded value of $1 at time 0 invested in the 1-period risk-free rate. Using the van Deventer [November 10, 2022] simulation, that money market fund takes on 500,000 values at each quarterly time step. To value deposits with lags, we can choose from (a) an extension of the closed-form solution from JMSVZ or (b) read the output from the van Deventer simulation and derive the valuation using lags and the parameters of the lags. In Appendix C, we give the example of the valuation of deposit interest in period 4 when the balance is constant and the deposit interest rate is a function of the current 1-period risk-free rate and its lagged values from 1 period and 2 periods before:

$$R_L(t+\Delta, t+4\Delta) = (\alpha_L + \beta_{L0}[r(t+3\Delta, t+4\Delta)] + \beta_{L1}[r(t+2\Delta, t+3\Delta)]$$

$$+\beta_{L2}[r(t+\Delta, t+2\Delta)] + e(t+4\Delta))$$

The notation is the same as in the JMSVZ paper. R_L is the deposit interest rate (uncompounded). The risk-free rate is designated r, which is also uncompounded. We assume the coefficients are derived from regression analysis which results in the error term e. The analyst can use either the analytical solution or derive the solution numerically using the regression coefficients and the valuation of deposit interest, deposit balances and their changes, and relevant non-interest expense using output from the van Deventer simulation or another simulation of Treasury yields.

2. How would the model be changed if the deposit rate and deposit balances were a function of both the current short-term risk-free rate but also the default probability of the issuing bank?

This question suggests a very valuable extension to the analysis that most practicing bankers ignore for a simple reason: their financial institution by definition exists because it has not gone bankrupt. This is a direct extension of Hilscher, Jarrow and van Deventer [2022], an analysis that gives an explicit valuation formula for corporate debt as a function of the risk-free yield curve, the term structure of default probabilities, and market-implied liquidity parameters and recovery rates. Deposits can be thought of as a form of corporate debt but with different recovery rates (because of deposit insurance) and liquidity premiums (which can be negative) that reflect the convenience of bank deposits to depositors. A valuation formula like that in Appendix B can be derived for many useful cases, but a numerical valuation using the existing Treasury simulation is probably the most efficient method for most practical applications.

There are two key points relevant to this question. First, bankers cannot rely on their own bank's history to do this analysis because (by definition) their bank has not failed. Second, banks do fail and deposit rates in the course of failure definitely have exceeded both Treasury yields and the certificate of deposit yields from competing but solvent institutions. To assume such an event can never happen to one's own bank is one of the greatest risk management mistakes one can make.

3. How would the "replicating portfolio" be calculated to hedge non-maturity deposit valuation risk if the deposit amounts are equally divided among constant starting balances with ultimate maturities from 1 period to N periods?

This methodology was popularized by Dr. Dennis Uyemura when he was head of asset and liability management for First Interstate Bancorp in Los Angeles in the 1980s. In the original JMSVZ paper, the valuation formula and hedge ratios were given for a non-maturity deposit of N periods, with N being any integer. Assume we have 40 91-day periods and the original deposit amount is $500 million for 40 different deposit categories, one for each of the 91-day maturities out to ten years. The JMSVZ paper gives hedge ratios for a non-maturity deposit with any assumed length. The hedge ratios for the first and last periods are unique, but the hedge ratios for periods 2 through N-1 are the same formula. Therefore, we calculate the hedges needed for the 1-period class of deposits, the 2-period class of deposits and so on out to the N-period class of deposits. The "replicating porfolio" is the sum of the hedges for all 40 deposit classes. We will be providing an example in the next paper on this topic.

4. How would "core deposits" be modeled as a component of total non-maturity deposits?

Janosi, Jarrow and Zullo (1999) presented an elegant continuous time formula to address this issue, but the JMSVZ paper makes the answer to this question very simple. If, in addition to the 40 deposit classes described in the answer to question 3, we have another $10 billion in "core deposits", we simply designate a new class, class 41 with a maturity of M periods. M can be whatever number of periods is considered "core," and the parameters for this class of core deposits can be different from the other 40 classes.

5. How does this model take into account things like the minimum positive balances for current accounts, the "minimum savings needs" of customers and the (historically) observed behavior of non-maturity deposits?

With respect to minimum positive balances for current accounts, this is a variation on question 4. The total of all minimum balances can be named "deposit class 42" with its own set of parameters. That approach runs the risk, however, that the depositor simply closes the account. The deposit balance equation would capture this impact.

With respect to the historically observed behavior of non-maturity deposits, this was hinted at in equation 1. We have assumed implicitly that historically observed behavior would be captured by the econometric formula for the deposit rates and the econometric formula for deposit balances. Valuation is very straightforward when the risk-free rate is an important variable in those two equations.

6. How do factors like inflation and unemployment rates come into play in the non-maturity deposit model?

They can be additional explanatory variables in the regressions described in equation 5. When it comes to valuation, however, the most straightforward closed form soliutions arise when inflation and unemployment are either (a) uncorrelated with the risk-free rate or (more likely) (b) linked to the risk-free rate in a simple way. Risk-neutral valuation, as Amin and Jarrow [1992] showed, involves taking the risk-neutral expected value of all cash flows, however determined, discounted by the risk-free money fund value over all scenarios.

2. Conclusion

The JMSVZ paper focuses on the discrete formulas for non-maturity deposit valuation orginally described in Jarrow and van Deventer [1996]. Because the discrete valuation formulas are straightforward to derive and extend, there are a wide array of approaches for which a direct closed form solution is available, as we show in Appendix B. When the deposit rate and balance equations become more elaborate, it is just as easy to read the output of a simulation of the risk-free yield curve, combine that output with the deposit parameters that matter, and calculate deposit values and hedge ratios. There is no need to generate the Treasury (or "risk-free") simulation more than once to do the analysis.

Please feel free to contact the author at info@kamakuraco.com with comments or questions.

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Appendix A from Jarrow, Melnikov, Sherman, van Deventer and Zorn (October 2022)

Introduction to Modern Multi-factor Interest Rate Valuation

It is convenient to express any analytical functions of the money fund in terms of observable zero-coupon bond prices whenever possible in order to minimize the need for simulation when assessing the risk of non-maturity deposits.

For any number of factors driving the yield curve and for any choice of factor volatility, the following for formulas are true as long as the Heath, Jarrow and Morton ("HJM") no arbitrage conditions are applied:

Valuation of $1 in j periods

$$E_t\left[\frac{1}{B(t,t+j\Delta)}\right] = P(t,t+j\Delta)$$

Valuation of the risk-free rate paid in j periods

$$E_t\left[\frac{r(t+(j-1)\Delta,t+j\Delta)}{B(t,t+j\Delta)}\right] = P(t,t+(j-1)\Delta) - P(t,t+j\Delta)$$

Valuation of the risk-free rate squared, paid in j periods

$$E_t\left[\frac{[r(t+(j-1)\Delta,t+j\Delta)]^2}{B(t,t+j\Delta)}\right]$$
$$= P(t,t+j\Delta) - 2P(t,t+(j-1)\Delta) + E_t\left[\frac{B(t+(j-1)\Delta,t+j\Delta)}{B(t,t+(j-1)\Delta)}\right]$$

The first condition is always true when HJM conditions apply. The second condition values a security which pays the unannualized risk free rate at the end of a given period of length Δ but no principal is paid. The derivation of the valuation of the risk-free rate paid in j periods is as follows:

$$E_t\left[\frac{r(t+(j-1)\Delta,t+j\Delta)}{B(t,t+j\Delta)}\right] = E_t\left[\frac{1+r(t+(j-1)\Delta,t+j\Delta)-1}{B(t,t+j\Delta)}\right]$$

$$E_t\left[\frac{r(t+(j-1)\Delta,t+j\Delta)}{B(t,t+j\Delta)}\right]$$
$$= E_t\left[\frac{B(t+(j-1)\Delta,t+j\Delta)}{B(t,t+(j-1)\Delta)B(t+(j-1)\Delta,t+j\Delta)}\right] - E_t\left[\frac{1}{B(t,t+j\Delta)}\right]$$

$$E_t\left[\frac{r(t+(j-1)\Delta, t+j\Delta)}{B(t,t+j\Delta)}\right] = P(t,t+(j-1)\Delta) - P(t,t+j\Delta)$$

The valuation of the square of the risk-free rate stems from modeling when the deposit balance, not just the deposit rate, is a function of the risk-free rate. The valuation can be derived as follows:

We note that

$$[1+r(t+(j-1)\Delta,t+j\Delta)][1-r(t+(j-1)\Delta,t+j\Delta)] = 1-[r(t+(j-1)\Delta,t+j\Delta)]^2$$

$$[r(t+(j-1)\Delta,t+j\Delta)]^2 = 1-[1+r(t+(j-1)\Delta,t+j\Delta)][1-r(t+(j-1)\Delta,t+j\Delta)]$$

That means that

$$E_t\left[\frac{[r(t+(j-1)\Delta,t+j\Delta)]^2}{B(t,t+2\Delta)}\right] = E_t\left[\frac{1-[1+r(t+(j-1)\Delta,t+j\Delta)][1-r(t+(j-1)\Delta,t+j\Delta)]}{B(t,t+j\Delta)}\right]$$

$$E_t\left[\frac{[r(t+(j-1)\Delta,t+j\Delta)]^2}{B(t,t+2\Delta)}\right] = P(t,t+j\Delta) - E_t\left[\frac{[1+r(t+(j-1)\Delta,t+j\Delta)][1-r(t+(j-1)\Delta,t+j\Delta)]}{B(t,t+(j-1)\Delta)B(t+(j-1)\Delta,t+j\Delta)}\right]$$

$$E_t\left[\frac{[r(t+(j-1)\Delta,t+j\Delta)]^2}{B(t,t+2\Delta)}\right] = P(t,t+j\Delta) - E_t\left[\frac{B(t+(j-1)\Delta,t+j\Delta)[1-r(t+(j-1)\Delta,t+j\Delta)]}{B(t,t+(j-1)\Delta)B(t+(j-1)\Delta,t+j\Delta)}\right]$$

$$E_t\left[\frac{[r(t+(j-1)\Delta,t+j\Delta)]^2}{B(t,t+2\Delta)}\right] = P(t,t+j\Delta) - E_t\left[\frac{[1-r(t+(j-1)\Delta,t+j\Delta)]}{B(t,t+(j-1)\Delta)}\right]$$

$$E_t\left[\frac{[r(t+(j-1)\Delta,t+j\Delta)]^2}{B(t,t+2\Delta)}\right] = P(t,t+j\Delta) - P(t,t+(j-1)\Delta) + E_t\left[\frac{r(t+(j-1)\Delta,t+j\Delta)}{B(t,t+(j-1)\Delta)}\right]$$

$$E_t\left[\frac{[r(t+(j-1)\Delta, t+j\Delta)]^2}{B(t,t+2\Delta)}\right]$$
$$= P(t,t+j\Delta) - P(t,t+(j-1)\Delta) + E_t\left[\frac{1+r(t+(j-1)\Delta, t+j\Delta)-1}{B(t,t+(j-1)\Delta)}\right]$$

$$E_t\left[\frac{[r(t+(j-1)\Delta, t+j\Delta)]^2}{B(t,t+2\Delta)}\right]$$
$$= P(t,t+j\Delta) - 2P(t,t+(j-1)\Delta) + E_t\left[\frac{B(t+(j-1)\Delta, t+j\Delta)}{B(t,t+(j-1)\Delta)}\right]$$

Appendix B: Valuing the Components in Deposit Cash Flows from Jarrow, Melnikov, Sherman, van Deventer and Zorn (October 2022)

Valuation of the Deposit in Period 1

In the first period, both the risk-free rate and the deposit balance are known with certainty. The value of the first period cash flows can be calculated using the relationships given above.

$$V(t,t+\Delta) = E_t\left[\frac{interest\ amount + noninterest\ expense + outgoing\ deposit\ balance}{B(t,t+\Delta)}\right]$$

$$V(t,t+\Delta)$$
$$= E_t\left[\frac{\big(\alpha_L + \beta_L[r(t,t+\Delta)] + e(t+\Delta)\big)D(t,t+\Delta) + a_0 + a_1 D(t,t+\Delta) + D(t,t+\Delta)}{B(t,t+\Delta)}\right]$$

Using the formulae in Appendix A, the deposit valuation formula can be calculated directly.

$$V(t,t+\Delta) = [a_0 + D(t,t+\Delta)(1+\alpha_L + a_1 - \beta_L)]P(t,t+\Delta) + D(t,t+\Delta)\beta_L$$

Valuation of the Forward Deposit in Period 2

The valuation of a 1-period forward deposit in period 2 when we assume the deposit balance in period 2 is a linear function of the risk-free rate in period 2 relies on a number of assumptions in the worked example presented in this note.

The deposit rate is assumed to be linear in the risk-free (unannualized) rate.

$$R_L(t+\Delta, t+2\Delta) = (\alpha_L + \beta_L[r(t+\Delta, t+2\Delta)] + e(t+2\Delta))$$

The deposit balance is also assumed to be linear in the risk-free rate.

$$D(t+\Delta, t+2\Delta) = d_0 + d_1[r(t+\Delta, t+2\Delta)] + d_e(t+2\Delta)$$

Noninterest expense associated with the deposit franchise is assumed to be linear in the deposit balance, which in turn is linear in the risk-free rate.

$$NIE(t+\Delta, t+2\Delta) = a_0 + a_1 D(t+\Delta, t+2\Delta)$$

$$NIE(t+\Delta, t+2\Delta) = a_0 + a_1[d_0 + d_1[r(t+\Delta, t+2\Delta)] + d_e(t+2\Delta)]$$

$$NIE(t+\Delta, t+2\Delta) = a_0 + a_1 d_0 + a_1 d_1 r(t+\Delta, t+2\Delta) + a_1 d_e(t+2\Delta)$$

The interest amount paid is the product of two functions linear in the risk-free rate, the deposit rate equation and the deposit balance equation.

$$interest = (\alpha_L + \beta_L[r(t+\Delta, t+2\Delta)] + e(t+2\Delta))(d_0 + d_1[r(t+\Delta, t+2\Delta)] + d_e(t+2\Delta)))$$

$$interest = d_0\alpha_L + [d_0\beta_L + \alpha_L d_1][r(t+\Delta, t+2\Delta)])$$

$$+d_1\beta_L[r(t+\Delta, t+2\Delta)]^2 + d_0 e(t+2\Delta) + d_1 e(t+2\Delta)[r(t+\Delta, t+2\Delta)])$$

$$+(d_e(t+2\Delta))(\alpha_L + \beta_L[r(t+\Delta, t+2\Delta)] + e(t+2\Delta))$$

Valuation of the Deposits in Period 2

The valuation of the 1-period forward deposit maturing at the end of period 2 is a function of the four components of deposit cash flows received or paid at two different points in time, the beginning of the period and the end of the period.

$$V_f(t, t+2\Delta) = -E_t\left[\frac{incoming\ deposit\ balance}{B(t, t+\Delta)}\right] + E_t\left[\frac{interest\ amount}{B(t, t+2\Delta)}\right] + E_t\left[\frac{interest\ amount}{B(t, t+2\Delta)}\right] + E_t\left[\frac{outgoing\ deposit\ balance}{B(t, t+2\Delta)}\right]$$

Using the formulae in Appendix A, the period 2 deposit can be valued using this equation.

$$V_f(t, t+2\Delta) = [a_0 + (d_0 - d_1)(1 + \alpha_L + a_1 - \beta_L) - d_1]P(t, t+2\Delta)$$
$$+ [d_0\beta_L + d_1(1 + \alpha_L + a_1 - 2\beta_L) - (d_0 + d_1)]P(t, t+\Delta)$$
$$- d_1(1 - \beta_L) + E_t\left[\frac{B(t+\Delta, t+2\Delta)}{B(t, t+\Delta)}\right]$$

The valuation formula for the period 3 deposit confirms the valuation pattern as maturity lengthens.

The valuation formula for the N-period deposit is the sum of the valuations for period 1, period 2, period 3 and so on.

$$V(t, t+N\Delta) = V(t, t+\Delta) + \sum_{i=2}^{N} V_f(t, t+i\Delta)$$

$$V(t, t+N\Delta) = K_1 P(t, t+\Delta) + \beta_L D(t, t+\Delta) + K_2 \sum_{i=2}^{N} P(t, t+i\Delta) + K_3 \sum_{i=1}^{N-1} P(t, t+i\Delta)$$
$$- K_4 \sum_{i=2}^{N} E_t[B(t, t+i\Delta)]$$

The four constants are defined as follows:

$$K_1 = [a_0 + D(t, t+\Delta)(1 + \alpha_L + a_1 - \beta_L)]$$

$$K_2 = (a_0 + [d_0 - d_1](1 + \alpha_L + a_1 - \beta_L))$$

$$K_3 = [d_1(2 + \alpha_L + a_1 - 2\beta_L) - d_0(1 - \beta_L)]$$

$$K_4 = d_1(1 - \beta_L)$$

Appendix C: Lagged Response of the Deposit Rate and Balance

In the base model, the deposit rate is assumed to be linear in the risk-free (unannualized) rate.

$$R_L(t+\Delta, t+2\Delta) = (\alpha_L + \beta_L[r(t+\Delta, t+2\Delta)] + e(t+2\Delta))$$

Now consider the case where the deposit rate is determined not only by the current risk-free rate but also by the two previous periods' risk-free rate. We number the beta coefficient with subscripts 0, 1 and 2 for 0 lag, 1 period lag, and two-period lag. The deposit rate equation for the second period becomes

$$R_L(t+\Delta, t+2\Delta) = (\alpha_L + \beta_{L0}[r(t+\Delta, t+2\Delta)] + \beta_{L1}[r(t, t+\Delta)]$$
$$+\beta_{L2}[r(t-\Delta, t)] + e(t+2\Delta))$$

The lagged rates are not random in period 2. All three rates become random when modeling the deposit interest in period 4. In the simplest case where the deposit balance is a constant D, the risk-neutral present value of the deposit interest amount is derived as follows for deposit interest paid at the end of period 4:

$$E_t\left[\frac{interest\ amount}{B(t, t+4\Delta)}\right]$$

$$= D\alpha_L E_t\left[\frac{1}{B(t, t+4\Delta)}\right] + D\beta_{L0} E_t\left[\frac{r(t+3\Delta, t+4\Delta)}{B(t, t+4\Delta)}\right] + D\beta_{L1} E_t\left[\frac{r(t+2\Delta, t+3\Delta)}{B(t, t+4\Delta)}\right]$$
$$+ D\beta_{L2} E_t\left[\frac{r(t+\Delta, t+2\Delta)}{B(t, t+4\Delta)}\right] + DE_t\left[\frac{e(t+4\Delta)}{B(t, t+4\Delta)}\right]$$

$$= D\alpha_L P(t, t+4\Delta) + D\beta_{L0}[P(t, t+3\Delta) - P(t, t+4\Delta)]$$
$$+ D\beta_{L1} E_t\left[\frac{1 + r(t+2\Delta, t+3\Delta) - 1}{B(t, t+3\Delta)} \frac{1}{B(t+3\Delta, t+4\Delta)}\right]$$
$$+ D\beta_{L2} E_t\left[\frac{1 + r(t+\Delta, t+2\Delta) - 1}{B(t, t+2\Delta)} \frac{1}{B(t+2\Delta, t+4\Delta)}\right]$$

$$= D\alpha_L P(t,t+4\Delta) + D\beta_{L0}[P(t,t+3\Delta) - P(t,t+4\Delta)] + D\beta_{L1}E_t\left[\left(\frac{1+r(t+2\Delta,t+3\Delta)-1}{B(t,t+3\Delta)}\right)\left(\frac{1}{B(t+3\Delta,t+4\Delta)}\right)\right] + D\beta_{L2}E_t\left[\left(\frac{1+r(t+\Delta,t+2\Delta)-1}{B(t,t+2\Delta)}\right)\left(\frac{1}{B(t+2\Delta,t+4\Delta)}\right)\right]$$

$$= D\alpha_L P(t,t+4\Delta) + D\beta_{L0}[P(t,t+3\Delta) - P(t,t+4\Delta)] + D\beta_{L1}E_t\left[\left(\frac{1}{B(t,t+2\Delta)} - \frac{1}{B(t,t+3\Delta)}\right)\left(\frac{1}{B(t+3\Delta,t+4\Delta)}\right)\right] + D\beta_{L2}E_t\left[\left(\frac{1}{B(t,t+\Delta)} - \frac{1}{B(t,t+2\Delta)}\right)\left(\frac{1}{B(t+2\Delta,t+4\Delta)}\right)\right]$$

$$= D\alpha_L P(t,t+4\Delta) + D\beta_{L0}[P(t,t+3\Delta) - P(t,t+4\Delta)] + D\beta_{L1}E_t\left[\left(\frac{B(t+2\Delta,t+3\Delta)}{B(t,t+4\Delta)} - \frac{1}{B(t,t+4\Delta)}\right)\right] + D\beta_{L2}E_t\left[\left(\frac{B(t+\Delta,t+2\Delta)}{B(t,t+4\Delta)} - \frac{1}{B(t,t+4\Delta)}\right)\right]$$

$$D\alpha_L P(t,t+4\Delta) + D\beta_{L0}[P(t,t+3\Delta) - P(t,t+4\Delta)] + D\beta_{L1}E_t\left[\frac{B(t+2\Delta,t+3\Delta)}{B(t,t+4\Delta)}\right] - D\beta_{L1}P(t,t+4\Delta) + D\beta_{L2}E_t\left[\frac{B(t+\Delta,t+2\Delta)}{B(t,t+4\Delta)}\right] - D\beta_{L1}P(t,t+4\Delta)$$

The time 0 value (cost) of deposit interest paid at the end of period 4 when deposit balances are constant takes this form:

$$D[\alpha_L - \beta_{L0} - \beta_{L1} - \beta_{L2}]P(t,t+4\Delta) + D\beta_{L0}P(t,t+3\Delta) + D\beta_{L1}E_t\left[\frac{B(t+2\Delta,t+3\Delta)}{B(t,t+4\Delta)}\right] + D\beta_{L2}E_t\left[\frac{B(t+\Delta,t+2\Delta)}{B(t,t+4\Delta)}\right]$$

The value of the stream of periodic interest paid on deposits when deposit interest is the function of the current risk-free rate and the risk-free rate lagged for one and two periods is a summation of the formula above for periods 1 through N.