Kamakura Corporation
2222 Kalakaua Avenue
Suite 1400
Honolulu, Hawaii 96815, USA
phone 1 808 791 9888
facsimile 1 808 791 9898
www.kamakuraco.com

A Practical Guide to the Valuation of Coupon-Bearing Fixed Income Securities

Robert A. Jarrow and Donald R. van Deventer

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Abstract

The purpose of this note is to illustrate the use of modern reduced form models for valuing defaultable coupon-bearing securities like risky sovereign bonds, corporate bonds, and retail loans like auto loans and mortgage loans. We focus on applications, not a review of the literature and not the derivations of the model. Section 1 of this note shows the valuation formulas for the reduced form bond model when there are no liquidity premiums in the marketplace. Section 2 examines three different assumptions about liquidity premiums and shows the implications of liquidity premiums for modeling recovery amounts. Section 3 discusses econometric procedures for deriving recovery rates and the associated liquidity parameters. Section 4 provides recent examples and summarizes related conclusions about implementation of the models using traded bond price data from TRACE.

The evolution of the reduced form bond model implementation techniques has proven that reduced form default probabilities, in combination with the valuation theory, provide greater accuracy in valuation than commonly used market conventions. Superior accuracy in valuation implies superior performance in credit spread generation, hedging, and performance attribution.

Introduction

The purpose of this short note is to illustrate the use of modern reduced form models for valuing defaultable coupon-bearing securities like risky sovereign bonds, corporate bonds, and retail loans like auto loans and mortgage loans. In this note, we focus on applications, not a review of the literature and not the derivations of the model. Hilscher, Jarrow and van Deventer [January 2022] provide a review of the literature and a formal derivation of one of the three implementations illustrated in this note. Jarrow and van Deventer [2018] show the model derivation in a discrete-time Heath, Jarrow, and Morton [1992] context.

Section 1 of this note shows the valuation formulas for the reduced form bond model when there are no liquidity premiums in the marketplace. Section 2 examines three different assumptions about liquidity premiums and shows the implications of liquidity premiums for modeling recovery amounts. Section 3 discusses econometric procedures for deriving recovery rates and the associated liquidity parameters. Section 4 provides recent examples and summarizes related conclusions about implementation of the models using traded bond price data from TRACE.

Section 1: Valuation when Liquidity Premiums are Zero

Traditionally, both practitioners and academics have preferred to derive bond valuations from an analysis of credit spreads. It turns out that it is much more accurate and efficient to derive bond valuation directly, and then to derive credit spreads from those valuations.

Why? Because credit spreads as conventionally defined are riddled with model error which obscures the valuation process. A quick summary of credit spread model errors is listed here:

- Credit spreads assume that principal payments and coupon payments have equal seniority, but that is false. There is no recovery on coupon payments, so coupon payments are junior to principal payments.
- Credit spreads, as conventionally defined, are calculated as the difference between the yield to maturity on a "comparable" Treasury bond and the yield to maturity on the corporate bond being studied.
    - Yield-to-maturity calculations assume the yield curve is flat
    - Yield-to-maturity differences among the bonds of the same issuer imply different discount rates for cash flow on the same payment date.
    - "Comparable" Treasury bond often means the nearest but shorter maturity "on the run" Treasury issue, not a true matched maturity Treasury

It is easier to avoid these false assumptions than to seek a half-successful work-around. That is what reduced form valuation does.

For the zero liquidity premium case, bonds can be valued as a portfolio of digital 0/1 coupon securities and digital 0/1 recovery securities. These "building block" securities can be described simply. The digital coupon security pays $1 if the bond's coupon security due at time T is paid on time, 0 otherwise. The digital recovery security pays $1 if the issuer of the bond defaults between current time t and time T, 0 otherwise. "Otherwise" includes the case in which the issuer defaults before time t or after time T, and it includes the case where the issuer never defaults.

We now introduce some simple notation:

$c(t,t+j\Delta)$	The value of coupon security which pays \$1 on the scheduled coupon payment date $t+j\Delta$ if default has not occurred by that time, \$0 otherwise
$r(t,t+j\Delta)$	The digital recovery security which pays \$1 at time $t+j\Delta$ if and only if default occurs at time T where $t+(j-1)\Delta<T<=t+j\Delta$

Now consider a coupon bearing bond that pays a dollar coupon amount K semi-annually and which matures in N periods. As discussed above, we analyze this bond via its components: N all-or-nothing coupon securities and N recovery securities. The security due in 2N periods is the all-or-nothing scheduled principal, which is indistinguishable in its characteristics (except for the amount) from the final coupon payment. We call this an "all-or-nothing" principal payment because recovery, if any, is modeled separately via the digital recovery securities. We can assume the recovery rate is either random or constant. For expositional ease, we assume the recovery rate is a known proportion δ of scheduled principal. The net present value of the bond, V, at time t is a function of these digital securities and the recovery rate δ:

$$V(t) = 100c(t, t + N\Delta) + K \sum_{i=1}^{N} c(t, t + i\Delta) + \delta 100 \sum_{i=1}^{N} r(t, t + i\Delta)$$

It is important to note that the total value paid by the buyer of a bond and received by the seller is “price” plus accrued interest. V(t) represents this net present value.

If the zero-coupon risk-free yield curve is observable, we can express V as a function of the related zero-coupon bond prices, adjusted for default risk. Using the usual set of HJM assumptions, Hilscher, Jarrow and van Deventer [2022] (“HJV”) show that the value of the coupon security for the first period is

$$c(t, t + \Delta) = P(t, t + \Delta)[1 - q(t + \Delta)]$$

The parameter q is the risk-neutral expected cumulative probability of default between time t and time t+Δ. The coupon security for the payment due in the ith period is similar:

$$c(t, t + i\Delta) = P(t, t + i\Delta)[1 - q(t + i\Delta)]$$

What is the value of the recovery security for period 1? Note that if an investor purchased one coupon security and one recovery security, both maturating at t+Δ, the investor receives $1 with certainty (and with no transactions costs in this section). Therefore, the sum of the values of the period 1 coupon and recovery securities must equal the value of a zero-coupon Treasury bill P:

$$c(t, t + \Delta) + r(t, t + \Delta) = P(t, t + \Delta)$$

Therefore, the value of the period 1 recovery security is a derivative security that depends on the value of the period 1 coupon security and the period 1 risk-free zero-coupon bond.

$$r(t, t + \Delta) = P(t, t + \Delta) - c(t, t + \Delta)$$

Using a similar no-arbitrage argument sequentially, the value of the recovery security for the 1 period interval ending at time t+iΔ is as follows:

$$r(t, t + i\Delta) = [1 - q(t + [i - 1]\Delta)]P(t, t + i\Delta) - c(t, t + i\Delta)$$

With these building block valuations in hand, bond valuation is a straightforward combination of risk-free zero yields and risk-neutral default probabilities. In the next section, we extend the model to allow for liquidity premiums in the bond market because of bid-offered spreads, other transactions costs, and the possibility of a “rescue” by government regulators as was often seen in the 2008-2010 crisis.

Section 2: Valuation with Non-Zero Liquidity Premiums

Three different liquidity assumptions have been used by the authors:

- Kamakura Risk Information Services ("KRIS") Version 1 Reduced Form Bond Model Assumptions
- HJV Reduced Form Bond Model Assumptions
- KRIS Version 2 Reduced Form Bond Model Assumptions

We discuss each in turn, in the order in which the 3 models have been implemented.

KRIS Version 1 Reduced Form Bond Model Assumptions

The liquidity discount function employed in this model was a simple 1-parameter discount function as a function of time to the relevant payment date:

$$Liquidity\ Discount = \exp\left[-(t + i\Delta - t)\beta_1\right]$$

This function is multiplied times the prior formula for the coupon security's value as follows:

$$c(t, t + i\Delta) = \exp\left[-(t + i\Delta - t)\beta_1\right]P(t, t + i\Delta)[1 - q(t + i\Delta)]$$

$$r(t, t + i\Delta) = [1 - q(t + [i - 1]\Delta)]P(t, t + i\Delta) - c(t, t + i\Delta)$$

The derivation of the recovery security continues to rely on the no-arbitrage relationship between the recovery and coupon securities. If liquidity discounts cause the coupon security to fall in value, the corresponding recovery security with rise in value. The parameter β_1 is constrained to be a positive number in this implementation.

Next, we turn to the HJV implementation.

HJV Reduced Form Bond Model Assumptions

In the HJV version, the liquidity discount function is the same, but it is applied to both the coupon securities and to the reduced form securities:

$$c(t, t + i\Delta) = \exp\left[-(t + i\Delta - t)\beta_1\right]P(t, t + i\Delta)[1 - q(t + i\Delta)]$$

$$r(t, t + i\Delta) = \exp\left[-(t + i\Delta - t)\beta_1\right][1 - q(t + [i - 1]\Delta)]P(t, t + i\Delta) - c(t, t + i\Delta)$$

In this specification, the parameter β_1 is still constrained to be a positive. The derivation of the implied recovery rate and implied beta are more complex in this specification, and the no-arbitrage relationship between recovery and coupon securities no longer applies.

KRIS Version 2 Reduced Form bond Model Assumptions

In the current (still in progress) KRIS implementation of Version 2 of the reduced form model, the liquidity discount function is generalized to a cubic function of the z years to payment date:

$$z = t + i\Delta - t$$

$$c(t, t + i\Delta) = \exp\left[\beta_0 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3\right]P(t, t + i\Delta)[1 - q(t + i\Delta)]$$

$$r(t, t+i\Delta) = \max\left(0, [1 - q(t + [i-1]\Delta)]P(t, t+i\Delta) - c(t, t+i\Delta)\right)$$

In this implementation, it is possible for liquidity "premiums" to be negative, as in the current case of Credit Suisse, whose bonds are trading well above the prices consistent with current default probabilities because of the potential rescue of the bank by Swiss authorities. Except for the case of negative liquidity premiums, the no arbitrage relationship between recovery securities and coupon securities is preserved. In the case of negative liquidity premiums, the value of each recovery security is constrained to be non-negative.

This version of the model can be shown to be the most accurate, as we demonstrate in the next section.

Section 3: Econometric Implementation

In each of the implementations above, non-linear least squares is employed to derive the beta values and recovery rate value that are most accurate. In every case, the recovery rate and beta parameters are derived from the regression parameters. Why? Functions that are bounded, like the recovery rate (which must be non-negative and less than or equal to one), must be calculated as functions with the same bounds and with continuous derivatives for any input values for efficient convergence of the non-linear least squares routine.

Similarly, the beta coefficients are derived so as to maximize the probability that the liquidity premium is the predominate positive premium. For example, it is helpful if β_0 is negative. One way to make this more likely is to make β_0 an exponential function of a regression coefficient. α_0. All three of the implementations make use of this kind of approach to some degree.

The result is a much better goodness of fit and reasonableness in implied parameter values. We turn to that discussion in Section 4.

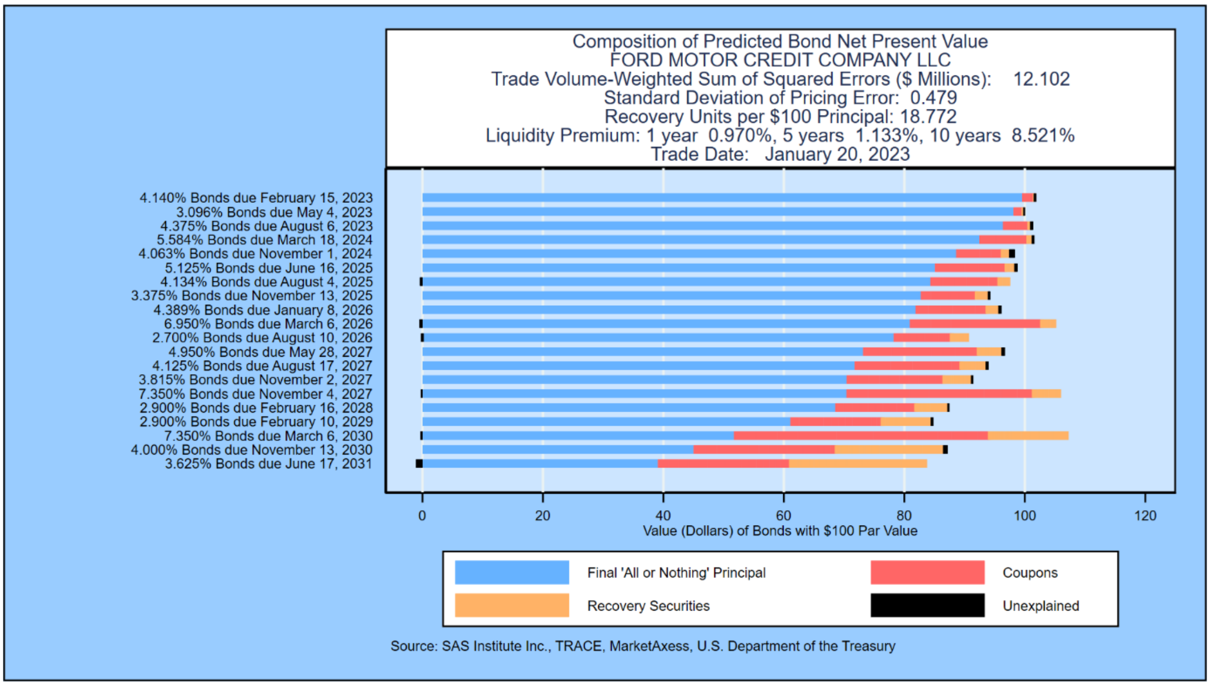
Section 4: Examples and Conclusions

Daily measures of goodness of fit for the KRIS Version 2.0 Reduced Form Model are provided at this link on the Kamakura Corporation website.

We use a ratings-related credit spread model of bond valuation as the challenger model in these model validation examples.

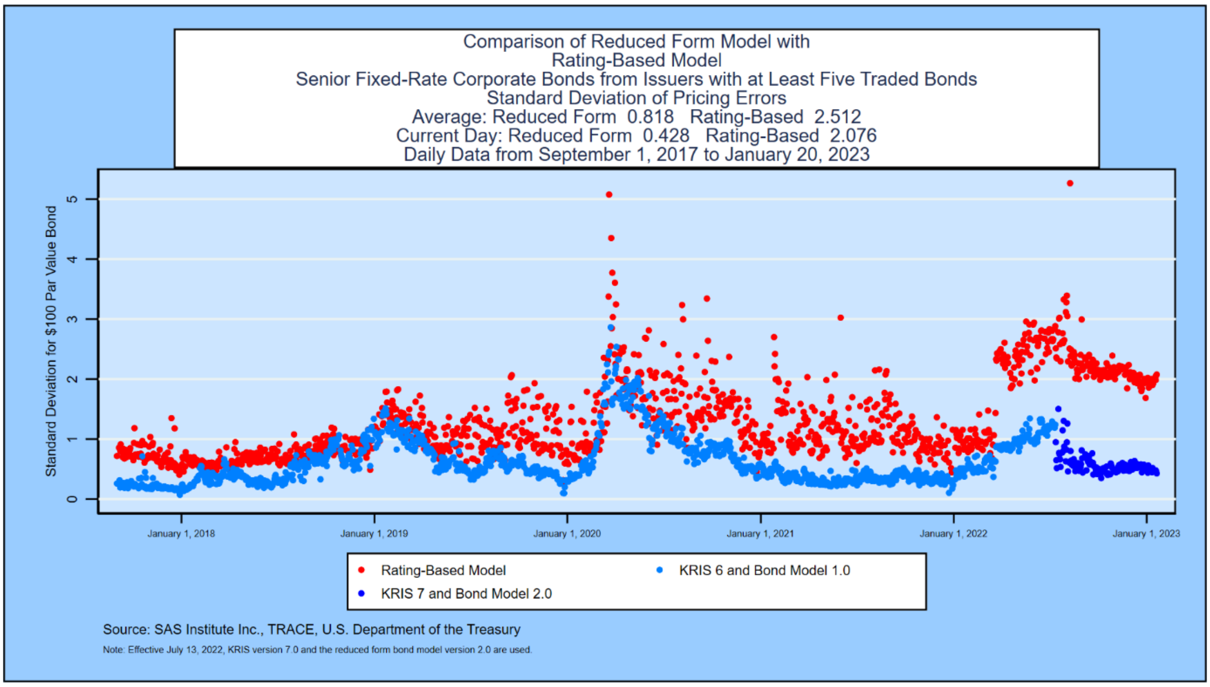
Exhibit 1 shows the decomposition of each bond's net present value into all-or-nothing principal, coupon securities value, and recovery securities value for Ford Motor Credit Corporation LLC on January 20, 2023. The title of the exhibit provides the recovery rate estimates and the standard deviation of net present value fitting errors. Implied liquidity premiums for 1, 5 and 10 years are also displayed.

Exhibit 1: Ford Motor Credit Company LLC Bond Parameters



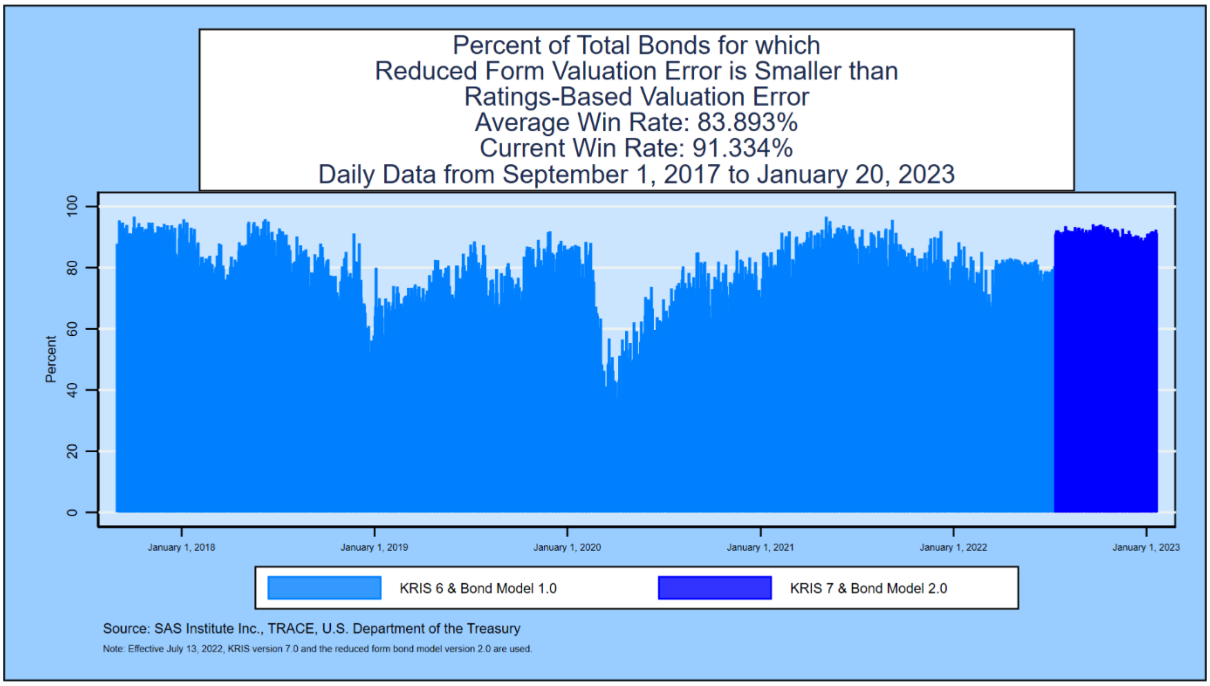
Since September 1, 2017, Kamakura Corporation has made a daily "challenger model" comparison for either KRIS Version 1 or KRIS Version 2 models versus the ratings-based credit spread model. Both versions are highly accurate, but the superiority of the KRIS Version 2 model is clearly shown on the right-hand side of Exhibit 2:

Exhibit 2: Time Series of Standard Deviations of Model Pricing Errors, 2017-2023



Using a sports analogy, one can employ a won/lost ratio approach to make model validation more intuitive. As with Exhibit 2, the improvement in accuracy in the KRIS Version 2 model is clear in Exhibit 3 as well.

Exhibit 3: Reduced Form Bond Model "Win Ratio" versus Ratings-Based Spread Model



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