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The Valuation and Hedging of Non-Maturity Deposits In Multi-Factor Interest Rate Models[1]

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Introduction

Financial institutions in all major sectors of the financial services business analyze and often hedge interest rate risk and other risks on a mark-to-market basis, as best practice financial economics prescribes. In commercial banking, however, one important liability type has proven problematic despite nearly 30 years of research on this liability: bank deposits with no explicit maturity. Deposit types which fall in the "non-maturity category" include demand deposits, many types of savings deposits, and a wide variety of unique deposit types that vary from country to country. This note extends previous modeling efforts to provide an explicit discrete valuation formula and framework that is valid for any number of factors driving the risk-free yield curve and for any no-arbitrage choice of interest rate volatilities, whether they be constant or stochastic, typically driven by the level of interest rates.

Description of Common Practice

Common practice for the valuation of non-maturity deposits has varied little over the last three decades, despite some changes in vocabularly during that time. Although financial theory dictates a mark-to-market approach to risk measurement and hedging, many bankers and regulators maintain a heavy focus on generally accepted accounting practice net income, and simulation over a one to two year horizon is a primary risk management focus. This risk management orientation is no different than the focus of the savings and loan associations in the United States in the 1980s and early 1990s. High interest rates devastated hundreds of savings institutions and the taxpayer bail-out cost was roughly one trillion dollars. Key aspects of common practice today include some or all of the following attributes:

- The primary focus of analysis is the impact of non-maturity deposits on net income and net interest income, not mark-to-market risk measurement.
- The main analytical calculation measures the link between the non-maturity deposit rate(s) and open market interest rates.
- This link to market rates is typically captured by a "deposit beta," which implies that only a single factor drives deposit rates.
- Deposit balances are handled in various ways:
  - Held constant
  - Varied over time in a deterministic way
  - Varied over time with some dependence on market interest rates
- The credit risk of the deposit-issuing bank is ignored as is the credit risk of competing banks and banks nationally.
- Non-interest expenses are ignored.

A critique of this portfolio of techniques is two-fold: (1) common practice does not take advantage of the advances in financial economics over the last five decades, and (2) the common practice techniques did not work for the savings and loan industry and there is no reason to believe that common practice will work any better in the future.

Why has common practice changed so little over the last three decades? A 40-year downward trend in interest rates and a very low level of interest rates over much of that period have lulled bankers into a false sense of security until recently. With the emergence of high inflation rates and the associated reaction from the Federal Reserve, non-maturity deposit rates are now headline news, as Exhibit 1 and Exhibit 2 from the Wall Street Journal illustrate.

Exhibit 1: Wall Street Journal, September 13, 2022

MARKETS | FINANCE

U.S. Banks Lost a Record $370 Billion in Deposits Last Quarter

The outflows will fuel a debate about how the Fed's inflation-calming moves are going to play out in the banking system

Exhibit 2: Wall Street Journal, October 3, 2022

PERSONAL FINANCE

Why Interest Rates Are Rising Everywhere—Except Your Savings Account

Many banks continue to offer meager yields on savings accounts, but it can pay off to shop around

We now turn to key milestones in research on the valuation and hedging of non-maturity deposits which provide the tools to deal with today's more complex deposit environment.

Review of the Literature

In the wake of the savings and loan crisis of the 1980s and 1990s, research on the valuation and interest rate risk of non-maturity deposits began in earnest. Hutchison and Pennachi [1996] analyzed the valuation and risk of non-maturity deposits with a 1-factor Vasicek model for the risk-free yield curve. Deposit balances were modeled as a random function of the risk-free rate, and non-interest expenses were explicitly included in the valuation. In the same year, Jarrow and van Deventer [1996] discussed the valuation of non-maturity deposits in a multi-factor Heath Jarrow and Morton context and illustrate valuation in the special case where the deposit rate and deposit balance are both linear functions of the 1-period risk-free rate. The valuation examples employ both the 1-factor Vasicek model and a multi-factor Heath, Jarrow and Morton model. The research was further extented in Jarrow and van Deventer [1998] and Janosi, Jarrow and Zullo [1999]. Kalkbrenner and Willing [2004] propose a 3-factor model of non-maturity deposit valuation. O'Brien [2000] uses a 1-factor Cox, Ingersoll and Ross model to value interest-bearing non-maturity deposits with an assymetric response in rising rate and falling rate environments. In last two decades following these papers, there has been a steady stream of general valuation approaches and valuation studies of unique non-maturity deposit structures.

We now turn to an extension of the Jarrow and van Deventer paper that offers explicit closed-form solutions for any N-factor no-arbitrage term structure model and any volatility structure. We also focus on the linear risk-free rate-driven deposit balances and deposit rate special case for expositional purposes.

Analytical Assumptions

The Heath, Jarrow and Morton framework is a completely general specification of how to benchmark and simulate risk-free interest rates in such a way that a Monte Carlo simulation will perfectly value observable market prices of risk-free securities at time zero. In addition, the HJM approach perfectly values the observable risk-free bonds for a holding period of any number of periods. It is important to note that many popular lattice structures based on one or two factor models fail that no arbitrage test.

In the same year that the main HJM article was published (but five years after it was originally written), Amin and Jarrow [1992] showed that the same multi-factor framework was the correct framework for valuing any securities which are traded but which are not subject to default risk.[2] Both papers allow the interest rate volatility to be constant or stochastic, most often driven by the relevant forward rates associated with the Jth risk factor. The discounting function is the compounded value of a risk-free "money market" fund that consists of the value of the money market function at the end of K time steps of arbitrary length over M scenarios in a simulation. While the money market fund is a fuction of the shortest risk-free rate, that rate and its movements are determined by multiple factors, the initial shape of the yield curve, and the volatility functions employed. Research by the authors shows that over a 13-country sample plus a consolidated "World" model, 7 to 15 factors are statistically significant.

The discrete formula for the money market fund is given here:

$$B(t,t+j\Delta) = [1+r(t,t+\Delta)][1+r(t+\Delta,t+2\Delta)][1+r(t+2\Delta,t+3\Delta)]\ldots[1+r(t+(j-1)\Delta,t+j\Delta)]$$

Valuation in the no arbitrage HJM framework discounts the cash flows at each time step and in each scenario by taking the average value of the cash flows using the "risk-neutral" probabilities associated with the instrument being valued.[3] The HJM no arbitrage conditions guarantee that the risk-neutral value of $1 over all time steps and maturities exactly matches the zero coupon bond prices of the initial yield curve. This formula allows an analyst to use either the observable zero-coupon bond prices or the simulated money fund values for valuation.

$$E_t\left[\frac{1}{B(t,t+j\Delta)}\right] = P(t,t+j\Delta)$$

We employ this capability extensively in this note.

Key Analytical Background

We summarize useful formulae that apply to all HJM discrete models with any number of factors and any type of interest rate volatility.

[2] Jarrow and Turnbull [1995] extended the framework to defaultable securities. Hilscher, Jarrow and van Deventer[January 2022] apply this framework to heavily traded corporate bonds and show a very high degree of pricing accuracy.
[3] Jarrow [2019] discusses the distinction between risk-neutral and observable or "empirical" probabilities in great detail in Chapter 16.

First we note that the money market fund function B can be broken into time segments which connect in a multiplicative way.

$$B(t,t+j\Delta) = B(t,t+(j-1)\Delta)B(t+(j-1)\Delta,t+j\Delta)$$

Using these money market fund time segments, we show in Appendix A that, for any number of factors and any no arbitrage volatility models, the value of a security which pays the risk-free rate of interest r but no principle payments is the difference in value between the zero coupon bond that matures at the start of the interest rate period and the zero coupon bond which matures at the end of the period.

$$E_t\left[\frac{r(t+(j-1)\Delta,t+j\Delta)}{B(t,t+j\Delta)}\right] = P(t,t+(j-1)\Delta) - P(t,t+j\Delta)$$

Finally, when valuing a security in which the amount and the interest rate are both functions of the risk-free rate r, it is convenient to value the square of the risk-free rate. We use this capability in the worked example below in the rate-driven deposit balance model:

$$E_t\left[\frac{[r(t+(j-1)\Delta,t+j\Delta)]^2}{B(t,t+j\Delta)}\right] = P(t,t+j\Delta) - 2P(t,t+(j-1)\Delta) + E_t\left[\frac{B(t+(j-1)\Delta,t+j\Delta)}{B(t,t+(j-1)\Delta)}\right]$$

The money market function appears in the last term of the valuation. This value is a by-product of any high quality HJM simulation of risk-free rates, as we show in the worked example below.

The derivation of these expressions is shown in Appendix A.

Valuing the Elements of Non-Maturity Deposit Cash Flow

Now we value a non-maturity deposit of N periods with a deposit balance that fluctuates as a function of the level of rates and where fixed and proportional noninterest expense functions are known.

The timing of cash flows is as follows:

Time t:

- The initial deposit amount D(t,t+Δ) has already been received.
- The initial deposit rate R_L(t,t+Δ) is known.
- The initial risk-free rate r(t,t+Δ) is known.

Time t+Δ:

- The risk-free rate r(t+Δ,t+2Δ) for period 2 becomes known.
- This determines the deposit rate for period 2, R_L(t+Δ,t+2Δ).
- The fixed noninterest expense a_0 for period 1 is paid.
- The proportional noninterest expense a_1D(t,t+Δ) for period 1 is paid
- The interest expense R_L(t,t+Δ)D(t,t+Δ) for period 1 is paid
- The prior period's deposit balance D(t,t+Δ) flows out.
- The period 2 random deposit balance D(t+Δ,t+2Δ) flows in.

Time t+2Δ:

- The risk-free rate r(t+2Δ,t+3Δ) for period 3 becomes known.
- This determines the deposit rate for period 3, R_L(t+2Δ,t+3Δ).
- The fixed noninterest expense a_0 for period 2 is paid.
- The proportional noninterest expense a_1D(t+Δ,t+2Δ) for period 2 is paid
- The interest expense R_L(t+Δ,t+2Δ) D(t+Δ,t+2Δ) for period 2 is paid
- The period 2's deposit balance D(t+Δ,t+2Δ) flows out.
- The period 3's random deposit balance D(t+2Δ,t+3Δ) flows in.

Time t+3Δ:

- The risk-free rate r(t+3Δ,t+4Δ) for period 4 becomes known.
- This determines the deposit rate for period 4, R(t+3Δ,t+4Δ).
- The fixed noninterest expense a_0 for period 3 is paid.
- The proportional noninterest expense a_1D(t+2Δ,t+3Δ) for period 3 is paid
- The interest expense R_L(t+2Δ,t+3Δ) D(t+2Δ,t+3Δ) for period 3 is paid
- The period 3 deposit balance D(t+2Δ,t+3Δ) flows out.
- The period 4 random deposit balance D(t+3Δ,t+4Δ) flows in.

This continues through the Nth period.

Valuation of the Deposits in Period 1

To illustrate the Amin and Jarrow valuation approach, we now turn to a special case in which the deposit balance is a linear function of the risk-free rate. We discuss more general formulations below. The parameters a_0, a_1, d_0, d_1, α_L and β_L are determined econometrically for these linear relationships in, for example, period 2:

The deposit rate is assumed to be linear in the risk-free (unannualized) rate.

$$R_L(t+\Delta, t+2\Delta) = (\alpha_L + \beta_L[r(t+\Delta, t+2\Delta)] + e(t+2\Delta))$$

The deposit balance is also assumed to be linear in the risk-free rate.

$$D(t+\Delta, t+2\Delta) = d_0 + d_1[r(t+\Delta, t+2\Delta)] + d_e(t+2\Delta)$$

For both of these linear relationships, we assume that the risk-neutral expected value of the error term is zero.

Noninterest expense associated with the deposit franchise is assumed to be linear in the deposit balance, which in turn is linear in the risk-free rate.

$$NIE(t+\Delta, t+2\Delta) = a_0 + a_1 D(t+\Delta, t+2\Delta)$$

In the first period, both the risk-free rate and the deposit balance are known with certainty. The value of the first period cash flows can be calculated using the relationships given above and explained in Appendix A and B.

$$V(t, t+\Delta) = E_t\left[\frac{interest\ amount + noninterest\ expense + outgoing\ deposit\ balance}{B(t, t+\Delta)}\right]$$

$$V(t, t+\Delta) = [a_0 + D(t, t+\Delta)(1 + \alpha_L + a_1 - \beta_L)]P(t, t+\Delta) + \beta_L D$$

Valuation of the Deposits in Period 2

In a similar way, the formulae of Appendix A are applied to the forward period 2 deposit valuation problem to value the four key components of cash flow:

$$V_f(t, t+2\Delta) = -E_t\left[\frac{incoming\ deposit\ balance}{B(t, t+\Delta)}\right] + \left[\frac{interest\ amount + noninterest\ expense + outgoing\ deposit\ balance}{B(t, t+2\Delta)}\right]$$

The valuation of the “forward” deposit in period 2 as of time t is given in Appendix B. It establishes the pattern used to derive the N-Period model.

Valuation of the Deposits from Period 1 to Period N

Finally, the value of the period 1 model is added to valuations for period 2 through N to get the general valuation formula when both the deposit rate and deposit balance are linear functions of the short rate. More general specifications are useful and convenient, but we restrict this note to linear functions to illustrate that there is no mystery to non-maturity deposit valuation and hedging.

$$V(t,t+N\Delta) = K_1 P(t,t+\Delta) + \beta_L D(t,t+\Delta) + K_2 \sum_{i=2}^{N} P(t,t+i\Delta) + K_3 \sum_{i=1}^{N-1} P(t,t+i\Delta) - K_4 \sum_{i=2}^{N} E_t\left[\frac{B(t+(i-1)\Delta, t+i\Delta)}{B(t,t+(i-1)\Delta)}\right]$$

The four constants K_1, K_2, K_3, and K_4 are defined as follows for the full model:

$$K_1 = [a_0 + D(t,t+\Delta)(1+\alpha_L + a_1 - \beta_L)]$$

$$K_2 = (a_0 + [d_0 - d_1](1+\alpha_L + a_1 - \beta_L))$$

$$K_3 = [d_1(2+\alpha_L + a_1 - 2\beta_L) - d_0(1-\beta_L)]$$

$$K_4 = d_1(1-\beta_L)$$

For the full model, the hedge ratios can be calculated using the four "K constants" given above. The hedge ratio with respect to 1 unit of non-maturity deposits using the zero-coupon bonds maturing in N periods is as follows:

$$\frac{\partial V(t,t+N\Delta)}{\partial P(t,t+N\Delta)} = K_2$$

For zero coupon bonds maturing in period i for i = 2 through N-1, the hedge ratios are

$$\frac{\partial V(t,t+N\Delta)}{\partial P(t,t+i\Delta)} = K_2 + K_3$$

For zero coupon bonds maturing in period 1,

$$\frac{\partial V(t,t+N\Delta)}{\partial P(t,t+\Delta)} = K_1 + K_3$$

The hedge ratio for the final term involves the expected money fund ratio. The hedge ratio is straightforward:

$$\frac{\partial V(t,t+N\Delta)}{\partial E_t\left[\frac{B(t+(i-1)\Delta, t+i\Delta)}{B(t,t+(i-1)\Delta)}\right]} = -K_4$$

Solutions for Special Cases

We now turn to special cases of this general formulation to show how dramatically hedge ratios can vary with what seem to be minor changes in assumptions. Those “minor changes” are in fact major changes.

Constant Deposit Balances with Zero Expenses that Pay the Risk-free Rate

$$V(t, t+N\Delta) = D(t, t+\Delta)$$

The relevant hedge ratio for this set of assumptions is zero, because the value doesn’t change as interest rate varies. Non-maturity deposits in this case are the equivalent of a Treasury bill-linked floating rate instrument.

The four constants K_1, K_2, K_3, and K_4 are defined as follows:

$$K_1 = 0$$

$$K_2 = 0$$

$$K_3 = 0$$

$$K_4 = 0$$

Constant Deposit Balances with Zero Expenses that Pay a Portion of the Risk-free Rate

Using the K coefficients below and the fact that the deposit balance is constant with d_0 = D(t,t+Δ), the value of non-maturity deposits as of time t is given by the following formula:

$$V(t, t+N\Delta) = \beta_L D(t, t+\Delta) + (D(t, t+\Delta)(1-\beta_L))P(t, t+N\Delta)$$

The hedge ratio with respect to 1 unit of non-maturity deposits using the zero-coupon bond maturing in N periods of length Δ is given as follows:

$$\frac{\partial V(t, t+N\Delta)}{\partial P(t, t+N\Delta)} = D(t, t+\Delta)(1-\beta_L)$$

The hedge ratio w is the amount of the N-period zero-coupon bond that makes the net change of a hedged portfolio equal to zero:

$$\frac{\partial \{V(t, t+N\Delta) + wP(t, t+N\Delta)\}}{\partial P(t, t+N\Delta)} = D(t, t+\Delta)(1-\beta_L) + w = 0$$

$$w = -D(t, t+\Delta)(1-\beta_L)$$

The hedge ratio for this model is to take a position in the N-period zero coupon bond in an amount proportional to the initial deposit with a ratio of 1-β_L. When the “deposit beta” is one, the hedge ratio is zero. The hedge ratio is for the full deposit amount when the deposit beta is zero. As we will see in the worked example below, both of these extremes are a poor approximation of reality.

The four constants K_1, K_2, K_3, and K_4 are defined as follows:

$$K_1 = D(t,t+\Delta)(1-\beta_L)$$

$$K_2 = D(t,t+\Delta)(1-\beta_L)$$

$$K_3 = -D(t,t+\Delta)(1-\beta_L)$$

$$K_4 = 0$$

Constant Deposit Balances with Zero Expenses that Pay a Constant Deposit Rate

$$V(t,t+N\Delta) = \alpha_L D(t,t+\Delta)\sum_{i=1}^{N} P(t,t+i\Delta) + D(t,t+\Delta)P(t,t+N\Delta)$$

The hedge ratio with respect to 1 unit of non-maturity deposits using the zero-coupon bonds maturing in 1 through N periods of length Δ are the same as they are with a coupon-bearing bond:

$$\frac{\partial V(t,t+N\Delta)}{\partial P(t,t+N\Delta)} = D(t,t+\Delta)(1+\alpha_L)$$

For zero coupon bonds maturing in period i for i = 1 through N-1, the hedge ratios are

$$\frac{\partial V(t,t+N\Delta)}{\partial P(t,t+i\Delta)} = \alpha_L D(t,t+\Delta)$$

The four constants K_1, K_2, K_3, and K_4 are defined in this case as follows, using the fact that d_0=D(t,t+Δ) :

$$K_1 = [D(t,t+\Delta)(1+\alpha_L)]$$

$$K_2 = [D(t,t+\Delta)(1+\alpha_L)]$$

$$K_3 = -D(t,t+\Delta)$$

$$K_4 = 0$$

Constant Deposit Balances with Zero Expenses that Pay a Fixed Rate Plus a Portion of the Risk-free Rate

$$V(t,t+N\Delta) = \beta_L D(t,t+\Delta) + D(t,t+\Delta)(1-\beta_L)P(t,t+N\Delta) + \alpha_L D(t,t+\Delta)\sum_{i=1}^{N} P(t,t+i\Delta)$$

The hedge ratio with respect to 1 unit of non-maturity deposits using the zero-coupon bonds maturing in N periods is as follows:

$$\frac{\partial V(t,t+N\Delta)}{\partial P(t,t+N\Delta)} = D(t,t+\Delta)(1+\alpha_L-\beta_L)$$

For zero coupon bonds maturing in period i for i = 1 through N-1, the hedge ratios are

$$\frac{\partial V(t,t+N\Delta)}{\partial P(t,t+i\Delta)} = \alpha_L D(t,t+\Delta)$$

The four constants K_1, K_2, K_3, and K_4 are defined as follows, again using the fact that d_0=D(t,t+Δ):

$$K_1 = [D(t,t+\Delta)(1+\alpha_L-\beta_L)]$$

$$K_2 = [D(t,t+\Delta)(1+\alpha_L-\beta_L)]$$

$$K_3 = [D(t,t+\Delta)\beta_L - D(t,t+\Delta)]$$

$$K_4 = 0$$

A Worked Example of Valuation in the Full Model

In this section, we use a 500,000 scenario simulation for U.S. Treasury yields 10 years forward using 91-day time steps. A detailed description of the simulation is available at this link. Exhibit 3 below incorporates a number of arbitrary but representative assumptions about parameter values to illustrate the calculation process. In practical application, these parameters would be estimated econometrically and updated regularly as more data is accumulated.

Valuations and deposit premiums are given in the table below. Hedge ratios can easily be calculated from the formulae above. Stress-testing of values and hedge ratios can also be done quicky by either (a) applying the desired yield shift to the simulated money market fund values from the simulation or (b) redoing the simulation using a shifted yield curve. Both methods have some advantages in practical use.

Exhibit 3

SAS Institute Inc.
HJM Simulation of U.S. Treasury Yield Curve
October 7, 2022
Number of Scenarios: 500,000
91-day Period Length out to 10 Years

Period Number	US Treasury Zero Coupon Bond Price	Money Market Factor	Deposit Net Present Value	Deposit Premium (Percent)
1	0.991436	0.000000	99.62	0.38
2	0.979932	1.004535	99.00	1.00
3	0.968716	0.991488	98.40	1.60
4	0.958869	0.979212	97.91	2.09
5	0.949373	0.967353	97.43	2.57
6	0.939458	0.955288	96.91	3.09
7	0.929041	0.958262	96.40	3.60
8	0.918428	0.935038	95.81	4.19
9	0.907996	0.948009	95.33	4.67
10	0.897976	0.918229	94.80	5.20
11	0.888437	0.883261	94.21	5.79
12	0.879346	0.878270	93.67	6.33
13	0.870630	0.865437	93.15	6.85
14	0.862212	0.881618	92.75	7.25
15	0.854029	0.867400	92.35	7.65
16	0.846029	0.840699	91.88	8.12
17	0.838171	0.850753	91.49	8.51
18	0.830422	0.818673	91.02	8.98
19	0.822760	0.864556	90.76	9.24
20	0.815170	0.749906	90.08	9.92
21	0.807647	0.788387	89.59	10.41
22	0.800190	0.768811	89.05	10.95
23	0.792807	0.811347	88.72	11.28
24	0.785507	0.858400	88.61	11.39
25	0.778302	0.808567	88.33	11.67
26	0.771204	0.766299	87.92	12.08
27	0.764223	0.741190	87.45	12.55
28	0.757369	0.827126	87.35	12.65
29	0.750643	0.724460	86.89	13.11
30	0.744047	0.764595	86.62	13.38
31	0.737573	0.883497	86.85	13.15
32	0.731213	0.854075	87.01	12.99
33	0.724954	0.799318	86.98	13.02
34	0.718782	0.641856	86.35	13.65
35	0.712679	0.786228	86.32	13.68
36	0.706629	0.632151	85.70	14.30
37	0.700613	0.758742	85.62	14.38
38	0.694615	0.650299	85.13	14.87
39	0.688618	0.644166	84.63	15.37
40	0.682607	0.669075	84.26	15.74

Assumptions

1. Risk-free rate is unannualized and expressed as a decimal
2. Initial deposit balance D= 100
3. Deposit rate RL = .00005+(.2)*r
4. Random deposit balance = 100+(-5)*r
5. Noninterest expenses = .25+(.0005)*random deposit balance.

Extensions to the Analysis

The worked example given in Exhibit 3 incorporates a number of additional features missing from common practice. There are useful extensions of Exhibit 3, many of which preserve closed form solutions, and all of which are easily merged with a simulation of Treasury yields to populate the table in Exhibit 3.

Some of the most important extensions are summarized here:

- Lagged adjustment of the deposit rates and balances to changes in the risk-free rate
- Assymetric responses of the deposit rates and balances to changes in the risk-free rate, distinguishing between rate increases and decreases
- Seasonal or other non-interest related variations in deposit balances
- Seasonal or other changes in model parameters
- Default risk of both the deposit issuer and the deposit issuer's competitors. The authors are not too young to remember major savings and loan associations in the mid-1980s paying more than Treasury yields and more than the certificates of deposit of credit-worthy banks even in the presence of deposit insurance. Needless to say, despite these high rates, deposits ran off to a degree that triggered government bail-outs.

Valuing the Option to Exit the Retail Deposit Business.

One of the distinguishing characteristics of non-maturity deposits is how difficult it is to vary the deposit rates by individual depositor characteristics. Varying rates by balance level or the longevity of the deposit are two of the options available to bankers and which can be incorporated in a granular extension of the analysis above.

Two of the most important options available to bankers are the options to (a) exit the existing retail business or (b) to enter such business. Both strategies have been used in the past. Option b is most often seen in the purchase of deposit franchises from banks that are troubled or which have other reasons for exiting the deposit business. Extensions of the analysis above can be employed in this "go or no go" decision.

Market-Implied Parameter Values

What parameters best fit market conditions at a given point in time? Two data sets embed a forecast of parameter values. One set is the market capitalization of banks with large deposit franchises. The other is the aging history of premiums paid in the acquisitions of deposit franchises sold by other banks, most of which took place during the savings and loan crisis. Using market capitalization-implied deposit valuations is not a new idea, as it was a regular part of selected large U.S commercial banks' asset and liability management practices in the mid 1980s. For additional details, please contact the authors.

Conclusions

Non-maturity deposits have a very large impact on both the actual and perceived interest rate sensitivity of deposit-gathering institutions. After four decades of generally declining interest rates, current market conditions definitely justify an improvement in the accuracy of the critical calculations used by financial institutions to measure and hedge the risk contribution of non-maturity deposits.

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Appendix A

Introduction to Modern Multi-factor Interest Rate Valuation

It is convenient to express any analytical functions of the money fund in terms of observable zero-coupon bond prices whenever possible in order to minimize the need for simulation when assessing the risk of non-maturity deposits.

For any number of factors driving the yield curve and for any choice of factor volatility, the following for formulas are true as long as the Heath, Jarrow and Morton ("HJM") no arbitrage conditions are applied:

Valuation of $1 in j periods

$$E_t\left[\frac{1}{B(t,t+j\Delta)}\right] = P(t,t+j\Delta)$$

Valuation of the risk-free rate paid in j periods

$$E_t\left[\frac{r(t+(j-1)\Delta,t+j\Delta)}{B(t,t+j\Delta)}\right] = P(t,t+(j-1)\Delta) - P(t,t+j\Delta)$$

Valuation of the risk-free rate squared, paid in j periods

$$E_t\left[\frac{[r(t+(j-1)\Delta,t+j\Delta)]^2}{B(t,t+j\Delta)}\right]$$
$$= P(t,t+j\Delta) - 2P(t,t+(j-1)\Delta) + E_t\left[\frac{B(t+(j-1)\Delta,t+j\Delta)}{B(t,t+(j-1)\Delta)}\right]$$

The first condition is always true when HJM conditions apply. The second condition values a security which pays the unannualized risk free rate at the end of a given period of length Δ but no principal is paid. The derivation of the valuation of the risk-free rate paid in j periods is as follows:

$$E_t\left[\frac{r(t+(j-1)\Delta,t+j\Delta)}{B(t,t+j\Delta)}\right] = E_t\left[\frac{1+r(t+(j-1)\Delta,t+j\Delta)-1}{B(t,t+j\Delta)}\right]$$

$$E_t\left[\frac{r(t+(j-1)\Delta,t+j\Delta)}{B(t,t+j\Delta)}\right]$$
$$= E_t\left[\frac{B(t+(j-1)\Delta,t+j\Delta)}{B(t,t+(j-1)\Delta)B(t+(j-1)\Delta,t+j\Delta)}\right] - E_t\left[\frac{1}{B(t,t+j\Delta)}\right]$$

$$E_t\left[\frac{r(t+(j-1)\Delta,t+j\Delta)}{B(t,t+j\Delta)}\right] = P(t,t+(j-1)\Delta) - P(t,t+j\Delta)$$

The valuation of the square of the risk-free rate stems from modeling when the deposit balance, not just the deposit rate, is a function of the risk-free rate. The valuation can be derived as follows:

We note that

$$[1 + r(t + (j-1)\Delta, t + j\Delta)][1 - r(t + (j-1)\Delta, t + j\Delta)] = 1 - [r(t + (j-1)\Delta, t + j\Delta)]^2$$

$$[r(t + (j-1)\Delta, t + j\Delta)]^2 = 1 - [1 + r(t + (j-1)\Delta, t + j\Delta)][1 - r(t + (j-1)\Delta, t + j\Delta)]$$

That means that

$$E_t\left[\frac{[r(t + (j-1)\Delta, t + j\Delta)]^2}{B(t, t + 2\Delta)}\right] = E_t\left[\frac{1 - [1 + r(t + (j-1)\Delta, t + j\Delta)][1 - r(t + (j-1)\Delta, t + j\Delta)]}{B(t, t + j\Delta)}\right]$$

$$E_t\left[\frac{[r(t + (j-1)\Delta, t + j\Delta)]^2}{B(t, t + 2\Delta)}\right] = P(t, t + j\Delta) - E_t\left[\frac{[1 + r(t + (j-1)\Delta, t + j\Delta)][1 - r(t + (j-1)\Delta, t + j\Delta)]}{B(t, t + (j-1)\Delta)B(t + (j-1)\Delta, t + j\Delta)}\right]$$

$$E_t\left[\frac{[r(t + (j-1)\Delta, t + j\Delta)]^2}{B(t, t + 2\Delta)}\right] = P(t, t + j\Delta) - E_t\left[\frac{B(t + (j-1)\Delta, t + j\Delta)[1 - r(t + (j-1)\Delta, t + j\Delta)]}{B(t, t + (j-1)\Delta)B(t + (j-1)\Delta, t + j\Delta)}\right]$$

$$E_t\left[\frac{[r(t + (j-1)\Delta, t + j\Delta)]^2}{B(t, t + 2\Delta)}\right] = P(t, t + j\Delta) - E_t\left[\frac{[1 - r(t + (j-1)\Delta, t + j\Delta)]}{B(t, t + (j-1)\Delta)}\right]$$

$$E_t\left[\frac{[r(t + (j-1)\Delta, t + j\Delta)]^2}{B(t, t + 2\Delta)}\right] = P(t, t + j\Delta) - P(t, t + (j-1)\Delta) + E_t\left[\frac{r(t + (j-1)\Delta, t + j\Delta)}{B(t, t + (j-1)\Delta)}\right]$$

$$E_t\left[\frac{[r(t + (j-1)\Delta, t + j\Delta)]^2}{B(t, t + 2\Delta)}\right] = P(t, t + j\Delta) - P(t, t + (j-1)\Delta) + E_t\left[\frac{1 + r(t + (j-1)\Delta, t + j\Delta) - 1}{B(t, t + (j-1)\Delta)}\right]$$

$$E_t\left[\frac{[r(t+(j-1)\Delta,t+j\Delta)]^2}{B(t,t+2\Delta)}\right]$$
$$= P(t,t+j\Delta) - 2P(t,t+(j-1)\Delta) + E_t\left[\frac{B(t+(j-1)\Delta,t+j\Delta)}{B(t,t+(j-1)\Delta)}\right]$$

Appendix B: Valuing the Components in Deposit Cash Flows

Valuation of the Deposit in Period 1

In the first period, both the risk-free rate and the deposit balance are known with certainty. The value of the first period cash flows can be calculated using the relationships given above.

$$V(t,t+\Delta) = E_t\left[\frac{interest\ amount + noninterest\ expense + outgoing\ deposit\ balance}{B(t,t+\Delta)}\right]$$

$$V(t,t+\Delta) = E_t\left[\frac{\left(\alpha_L + \beta_L[r(t,t+\Delta)] + e(t+\Delta)\right)D(t,t+\Delta) + a_0 + a_1 D(t,t+\Delta) + D(t,t+\Delta)}{B(t,t+\Delta)}\right]$$

Using the formulae in Appendix A, the deposit valuation formula can be calculated directly.

$$V(t,t+\Delta) = [a_0 + D(t,t+\Delta)(1+\alpha_L + a_1 - \beta_L)]P(t,t+\Delta) + D(t,t+\Delta)\beta_L$$

Valuation of the Forward Deposit in Period 2

The valuation of a 1-period forward deposit in period 2 when we assume the deposit balance in period 2 is a linear function of the risk-free rate in period 2 relies on a number of assumptions in the worked example presented in this note.

The deposit rate is assumed to be linear in the risk-free (unannualized) rate.

$$R_L(t+\Delta,t+2\Delta) = (\alpha_L + \beta_L[r(t+\Delta,t+2\Delta)] + e(t+2\Delta))$$

The deposit balance is also assumed to be linear in the risk-free rate.

$$D(t+\Delta,t+2\Delta) = d_0 + d_1[r(t+\Delta,t+2\Delta)] + d_e(t+2\Delta)$$

Noninterest expense associated with the deposit franchise is assumed to be linear in the deposit balance, which in turn is linear in the risk-free rate.

$$NIE(t+\Delta,t+2\Delta) = a_0 + a_1 D(t+\Delta,t+2\Delta)$$

$$NIE(t+\Delta,t+2\Delta) = a_0 + a_1[d_0 + d_1[r(t+\Delta,t+2\Delta)] + d_e(t+2\Delta)]$$

$$NIE(t+\Delta,t+2\Delta) = a_0 + a_1 d_0 + a_1 d_1 r(t+\Delta,t+2\Delta) + a_1 d_e(t+2\Delta)$$

The interest amount paid is the product of two functions linear in the risk-free rate, the deposit rate equation and the deposit balance equation.

$$\begin{aligned} interest = (\alpha_L + \beta_L[r(t+\Delta,t+2\Delta)] + e(t+2\Delta))(d_0 + d_1[r(t+\Delta,t+2\Delta)] \\ + d_e(t+2\Delta))) \end{aligned}$$

$$interest = d_0\alpha_L + [d_0\beta_L + \alpha_L d_1][r(t+\Delta,t+2\Delta)])$$

$$+d_1\beta_L[r(t+\Delta,t+2\Delta)]^2 + d_0 e(t+2\Delta) + d_1 e(t+2\Delta)[r(t+\Delta,t+2\Delta)])$$

$$+(d_e(t+2\Delta))(\alpha_L + \beta_L[r(t+\Delta,t+2\Delta)] + e(t+2\Delta))$$

Valuation of the Deposits in Period 2

The valuation of the 1-period forward deposit maturing at the end of period 2 is a function of the four components of deposit cash flows received or paid at two different points in time, the beginning of the period and the end of the period.

$$\begin{aligned} V_f(t,t+2\Delta) = -E_t\left[\frac{incoming\ deposit\ balance}{B(t,t+\Delta)}\right] + E_t\left[\frac{interest\ amount}{B(t,t+2\Delta)}\right] \\ + E_t\left[\frac{interest\ amount}{B(t,t+2\Delta)}\right] + E_t\left[\frac{outgoing\ deposit\ balance}{B(t,t+2\Delta)}\right] \end{aligned}$$

Using the formulae in Appendix A, the period 2 deposit can be valued using this equation.

$$\begin{aligned} V_f(t,t+2\Delta) = [a_0 + (d_0 - d_1)(1 + \alpha_L + a_1 - \beta_L) - d_1]P(t,t+2\Delta) \\ + [d_0\beta_L + d_1(1 + \alpha_L + a_1 - 2\beta_L) - (d_0 + d_1)]P(t,t+\Delta) \\ - d_1(1-\beta_L) + E_t\left[\frac{B(t+\Delta,t+2\Delta)}{B(t,t+\Delta)}\right] \end{aligned}$$

The valuation formula for the period 3 deposit confirms the valuation pattern as maturity lengthens. The valuation formula for the N-period deposit is the sum of the valuations for period 1, period 2, period 3 and so on.

$$V(t,t+N\Delta) = V(t,t+\Delta) + \sum_{i=2}^{N} V_f(t,t+i\Delta)$$

$$V(t,t+N\Delta) = K_1P(t,t+\Delta) + \beta_L D(t,t+\Delta) + K_2\sum_{i=2}^{N} P(t,t+i\Delta) + K_3\sum_{i=1}^{N-1} P(t,t+i\Delta) - K_4\sum_{i=2}^{N} E_t[B(t,t+i\Delta)]$$

The four constants are defined as follows:

$$K_1 = [a_0 + D(t,t+\Delta)(1+\alpha_L + a_1 - \beta_L)]$$

$$K_2 = (a_0 + [d_0 - d_1](1+\alpha_L + a_1 - \beta_L))$$

$$K_3 = [d_1(2+\alpha_L + a_1 - 2\beta_L) - d_0(1-\beta_L)]$$

$$K_4 = d_1(1-\beta_L)$$