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**A 12-Factor Heath, Jarrow, and Morton Stochastic Volatility Model
For 13-Country `World' Government Bonds,
Using Daily Data from January 1, 1962 through September 30, 2021
Donald R. van Deventer[1]
First Version: October 19, 2021
This Version: October 21, 2021**

ABSTRACT

Please note: Kamakura Corporation term structure models are updated monthly. For the most recent set of coefficients, contact info@kamakuraco.com

This paper analyzes the number and the nature of factors driving the movements in government yield curves from 13 countries from January 2, 1962 through September 30, 2021. The countries included are Australia, Canada, France, Germany, Italy, Japan, Russia, Singapore, Spain, Sweden, Thailand, the United Kingdom, and the United States. There are four primary reasons for constructing a composite "World" term structure model. First, as shown in the paper, the vast majority of the 13 countries do not have the same breadth (years to maturity span) and depth (length of the time series) as a consolidated model. Second, the spread of negative interest rates in many government markets is crucial data for those countries with no negative rate experience. Third, even if all countries had an equal amount of data, Stein's paradox and modern empirical Bayes approaches prove that such a consolidated model will produce more accurate results than the stand-alone single country models. Finally, the consolidated data set is six times larger than the U.S. Treasury history, with more than 90,000 observations. This large data set results in simulated yield curves that are more realistic in their shapes and in their evolution than a smaller data set can produce.

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A 12-Factor Heath, Jarrow, and Morton Stochastic Volatility Model
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This paper analyzes the number and the nature of factors driving the movements in government yield curves from 13 countries from January 2, 1962 through September 30, 2021. The countries included are Australia, Canada, France, Germany, Italy, Japan, Russia, Singapore, Spain, Sweden, Thailand, the United Kingdom, and the United States. This list of countries is composed of 11 members of the G20 group plus Sweden and Thailand.

There are four primary reasons for constructing a composite "World" term structure model. First, as shown in the paper, the vast majority of the 13 countries do not have the same breadth (years to maturity span) and depth (length of the time series) as a consolidated model. Second, the spread of negative interest rates in many government markets is crucial data for those countries with no negative rate experience. Third, even if all countries had an equal amount of data, Stein's paradox and modern empirical Bayes approaches prove that such a consolidated model will produce more accurate results than a stand-alone single country model. Finally, the consolidated data set is six times larger than the U.S. Treasury history, with more than 90,000 observations. This large data set results in simulated yield curves that are more realistic in their shapes and in their evolution than a smaller data set can produce.

Section I discusses the origin and characteristics of the daily data base of government yields from the 13-country data set. We discuss yield curve smoothness and volatility fitting as measures for judging the quality of government-generated yield curve time series and present an example of the technique in this paper. We conclude that the 13-country data set is an exceptionally rich collection of yield curves with negative rates. Significant cleansing of the database was necessary, particularly for data from Japan, Sweden, Spain, and Thailand.

Section II outlines the process for determining whether the interest rate volatility functions for the factors driving the 13-country yield curves are constants (an "affine" model) or stochastic, typically expressed as a function of the level of interest rates. Section III describes the process of fitting five different Heath, Jarrow, and Morton models to the 13-country yield data: models with 1, 2, 3, 6 and 12 factors. We conclude Section III with extensive Bayesian model validation procedures based on a 30-year forward-looking simulation of 500,000 scenarios, using both World and single-country parameters, for the U.S. Treasury curve on October 15, 2021. Section IV concludes the paper. Appendix A illustrates a sample model validation process for widely used one-factor term structure models using World data. Appendix B includes an analysis of empirical drift and stress tests of rate volatility. Appendix C compares the span of interest rates by level and months to maturity for the World data set versus the 13 countries which contributed data to the consolidated yield data base.

I. 13-Country Yield Data: Special Characteristics

A multi-factor term structure model is the foundation for best practice asset and liability management, market risk, economic capital, interest rate risk in the banking book, stress-

testing, and the internal capital adequacy assessment process. The objective in this paper is to illustrate the derivation of a multi-factor Heath Jarrow and Morton model of the yield curve data from 13 countries. As a by-product, the analysis has the potential to detect common data problems associated with yield curve histories and employs a standard methodology for quantification and resolution of those problems. Previous implementations of multi-factor Heath, Jarrow, and Morton models have covered the following bond market sectors:

Australia	Commonwealth Government Securities
Canada	Government of Canada Securities
France	French Government Securities
Germany	German Bunds
Italy	Italian Government Securities
Japan	Japanese Government Bonds
Russia	Russian Government Securities
Singapore	Singapore Government Securities
Spain	Spanish Government Bonds
Sweden	Swedish Government Securities
Thailand	Thai Government Securities
United Kingdom	United Kingdom Government Bonds
United States	U.S. Treasury Securities

The first step in data model validation for the 13-country data set is to examine the historical availability of bond yields over time. This availability is summarized in Table I. Because there are more than 200 different data regimes in the data set, the table is broken into 7 sub tables:

Table I

Kamakura Corporation
HJM 12 Factor Model
World Government
Using Daily Data from January 2, 1962 through September 30, 2021
Date of Analysis: September 30, 2021
Number of Observations by Data Regime

Data Regime	Start Date	End Date	Number of Observations
USA: 1, 3, 5 and 10 years	1/2/1962	6/30/1969	1,870
USA: 1, 3, 5, 7 and 10 years	7/1/1969	5/28/1976	1,722
USA: 1, 2, 3, 5, 7 and 10 years	6/1/1976	2/14/1977	178
USA: 1, 2, 3, 5, 7, 10 and 30 years	2/15/1977	12/31/1981	1,213
USA: 3 and 6 months with 1, 2, 3, 5, 7, 10 and 30 years	1/4/1982	9/30/1993	2,935
USA: 3 and 6 months with 1, 2, 3, 5, 7, 10, 20 and 30 years	10/1/1993	7/30/2001	1,960
USA: 1, 3 and 6 months with 1, 2, 3, 5, 7, 10, 20 and 30 years	7/31/2001	2/15/2002	135
USA: 1, 3 and 6 months with 1, 2, 3, 5, 7, 10, and 20 years	2/19/2002	2/8/2006	994
USA: Second era: 1, 3 and 6 months with 1, 2, 3, 5, 7, 10, 20 and 30 years	2/9/2006	3/29/2018	3,041
USA: 1 day, 1, 3 and 6 months with 1, 2, 3, 5, 7, 10, 20 and 30 years	4/2/2018	10/15/2018	137
USA: 1 day, 1, 2, 3, and 6 months with 1, 2, 3, 5, 7, 10, 20 and 30 years	10/16/2018	9/30/2021	740
aus: BBG-Yields to 10 Years	1/2/1996	12/31/2008	3,368
aus: RBA-Yields to 10 Years	1/2/2009	6/30/2021	3,153
can: Yields to 10 Years	1/2/2001	8/26/2021	5,171
deu: Yields to 30 years	8/7/1997	9/30/2021	6,058
esp: Yields to 15 Years	7/1/1987	1/4/2021	8,422
fra: months 1 3 6 9 12 years 2 5 10 30	1/2/2015	7/30/2021	1,673
gbr: Yields to 20 Years	1/2/1979	5/23/1979	99
gbr: Yields to 15 Years	5/24/1979	6/24/1980	272
gbr: Yields to 20 Years	6/25/1980	7/17/1980	17
gbr: Yields to 15 Years	7/18/1980	1/14/1985	1,134
gbr: Yields to 20 Years	1/15/1985	9/24/1985	176
gbr: Yields to 15 Years	9/25/1985	6/6/1986	176
gbr: Yields to 20 Years	6/9/1986	7/16/1991	1,292
gbr: Yields to 15 Years	7/17/1991	2/10/1992	145
gbr: Yields to 20 Years	2/11/1992	4/29/1992	55
gbr: Yields to 25 Years	4/30/1992	8/27/1992	84
gbr: Yields to 20 Years	8/28/1992	2/27/1996	885
gbr: Yields to 25 Years	2/28/1996	6/12/1996	72
gbr: Yields to 20 Years	6/13/1996	1/28/1998	413
gbr: Yields to 25 Years	1/29/1998	12/31/2015	4,529
gbr: Yields to 40 Years	1/4/2016	6/30/2021	1,389

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Data Regime	Start Date	End Date	Number of Observations
ita: Yields to 30 Years	5/21/2015	7/30/2021	1,607
jpn: 1-9 years	9/24/1974	5/19/1978	898
jpn: 4-9 years	5/22/1978	8/21/1978	66
jpn: 3-9 years	8/22/1978	8/21/1979	247
jpn: 2-9 years	8/22/1979	8/21/1980	249
jpn: 1-9 years, second era	8/22/1980	7/4/1986	1,444
jpn: 1-10 years	7/7/1986	11/28/1986	100
jpn: 1-10 years, 20 years	12/1/1986	8/29/1991	1,167
jpn: 1-10 years, 15, 20 years	8/30/1991	9/1/1999	1,969
jpn: 1-10 years, 15, 20, 30 years	9/2/1999	3/19/2004	1,114
jpn: 1-10 years, 15, 20, 25, 30 years	3/22/2004	11/5/2007	893
jpn: 1-10 years, 15, 20, 25, 30, 40 years	11/6/2007	3/31/2021	3,260
rus: Yields to 30 Years	1/8/2003	8/27/2021	4,626
sgp: day 1 month 3 years 1 2 5 7	9/1/1998	2/23/1999	120
sgp: day 1 month 3 years 1 2 5 7 10	2/24/1999	4/23/2002	796
sgp: day 1 month 3 years 1 2 5 7 10 15	4/24/2002	10/17/2007	1,379
sgp: day 1 month 3 years 1 2 5 7 10 15 20	10/18/2007	11/20/2012	1,279
sgp: day 1 month 3 years 1 2 5 7 10 15 20 30	11/21/2012	3/5/2013	71
sgp: day 1 month 3 6 years 1 2 5 7 10 15 20 30	3/6/2013	5/15/2014	300
sgp: day 1 month 6 years 1 2 5 7 10 15 20 30	5/16/2014	8/26/2014	72
sgp: month 6 years 1 2 5 7 10 15 20 30	8/27/2014	2/18/2015	122
sgp: years 1 2 5 7 10 15 20 30	2/23/2015	6/1/2021	1,572
swe: months 1 3 6 12 years 2 5 7 10	1/8/1987	1/28/1987	15
swe: months 1 3 6 years 2 5 7 10	1/29/1987	1/30/1987	2
swe: months 1 3 6 12 years 2 5 7 10	2/2/1987	3/30/1987	41
swe: months 1 3 6 years 2 5 7 10	3/31/1987	4/13/1987	10
swe: months 1 3 6 12 years 2 5 7 10	4/14/1987	1/22/1990	698
swe: months 1 3 6 12 years	1/23/1990	1/24/1990	2
swe: months 1 3 6 12 years 2 5 7 10	2/19/1990	8/29/1990	132
swe: months 3 6 12 years 2 5 7 10	8/30/1990	8/31/1990	2
swe: months 1 3 6 12 years 2 5 7 10	9/3/1990	11/29/1990	63
swe: months 3 6 12 years 2 5 7 10	11/30/1990	11/30/1990	1

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swe: months 1 3 6 12 years 2 5 7 10	12/3/1990	11/28/1994	977
swe: months 1 3 6 12 years 5 7 10	11/29/1994	11/30/1994	2
swe: months 1 3 6 12 years 2 5 7 10	12/1/1994	12/28/1994	19
swe: months 1 3 6 12 years 2 5 7	12/29/1994	12/30/1994	2
swe: months 1 3 6 12 years 2 5 7 10	1/2/1995	11/30/1995	232
swe: months 1 3 6 12 years 2 5 10	12/1/1995	12/1/1995	1
swe: months 1 3 6 12 years 2 5 7 10	12/4/1995	7/16/1997	402
swe: months 3 6 12 years 2 5 7 10	7/17/1997	7/29/1997	9
swe: months 1 3 6 12 years 2 5 7 10	7/30/1997	9/19/1997	38
swe: months 3 6 12 years 2 5 7 10	9/22/1997	9/26/1997	5
swe: months 1 3 6 12 years 2 5 7 10	9/29/1997	11/26/1997	43
swe: months 1 3 6 years 2 5 7 10	11/27/1997	11/28/1997	2
swe: months 1 3 6 12 years 2 5 7 10	12/1/1997	12/18/1997	14
swe: months 3 6 12 years 2 5 7 10	12/19/1997	12/23/1997	3
swe: months 1 3 6 12 years 2 5 7 10	12/29/1997	1/2/1998	3
swe: months 1 3 6 12 years 2 5 10	1/5/1998	1/5/1998	1
swe: months 1 3 6 12 years 2 5 7 10	1/7/1998	2/25/1998	36
swe: months 1 3 6 years 2 5 7 10	2/26/1998	2/27/1998	2
swe: months 1 3 6 12 years 2 5 7 10	3/2/1998	5/27/1998	59
swe: months 1 3 6 years 2 5 7 10	5/28/1998	5/29/1998	2
swe: months 1 3 6 12 years 2 5 7 10	6/2/1998	11/26/1998	127
swe: months 1 3 6 years 2 5 7 10	11/27/1998	11/30/1998	2
swe: months 1 3 6 12 years 2 5 7 10	12/1/1998	12/14/1998	10
swe: months 1 3 6 years 2 5 7 10	12/15/1998	12/16/1998	2
swe: months 1 3 6 12 years 2 5 7 10	12/17/1998	5/27/1999	107
swe: months 1 3 6 years 2 5 7 10	5/28/1999	5/31/1999	2
swe: months 1 3 6 12 years 2 5 7 10	6/1/1999	6/28/1999	19
swe: months 1 3 6 years 2 5 7 10	6/29/1999	6/30/1999	2
swe: months 1 3 6 12 years 2 5 7 10	7/1/1999	8/27/1999	42
swe: months 1 3 6 years 2 5 7 10	8/30/1999	8/31/1999	2
swe: months 1 3 6 12 years 2 5 7 10	9/1/1999	11/26/1999	63
swe: months 1 3 6 years 2 5 7 10	11/29/1999	11/30/1999	2

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swe: months 1 3 6 12 years 2 5 7 10	12/1/1999	4/3/2000	86
swe: months 1 3 12 years 2 5 7 10	4/4/2000	4/4/2000	1
swe: months 1 3 6 12 years 2 5 7 10	4/5/2000	6/26/2000	53
swe: months 3 6 12 years 2 5 7 10	6/27/2000	6/27/2000	1
swe: months 1 3 6 12 years 2 5 7 10	6/28/2000	7/11/2000	10
swe: months 3 6 12 years 2 5 7 10	7/12/2000	7/12/2000	1
swe: months 1 3 6 12 years 2 5 7 10	7/13/2000	12/21/2001	368
swe: months 3 6 12 years 2 5 7 10	12/27/2001	1/29/2002	22
swe: months 1 3 6 12 years 2 5 7 10	1/30/2002	8/28/2002	145
swe: months 1 3 6 years 2 5 7 10	8/29/2002	8/30/2002	2
swe: months 1 3 6 12 years 2 5 7 10	9/2/2002	10/28/2002	41
swe: months 1 3 6 years 2 5 7 10	10/29/2002	10/29/2002	1
swe: months 1 3 6 12 years 2 5 7 10	10/30/2002	11/27/2002	21
swe: months 1 3 6 years 2 5 7 10	11/28/2002	11/29/2002	2
swe: months 1 3 6 12 years 2 5 7 10	12/2/2002	8/27/2003	181
swe: months 1 3 6 years 2 5 7 10	8/28/2003	8/29/2003	2
swe: months 1 3 6 12 years 2 5 7 10	9/1/2003	6/18/2004	200
swe: months 1 6 12 years 2 5 7 10	6/21/2004	6/28/2004	5
swe: months 1 3 6 12 years 2 5 7 10	6/29/2004	6/4/2009	1,243
swe: months 1 3 6 years 2 5 7 10	6/5/2009	12/17/2009	139
swe: months 1 3 6 12 years 2 5 7 10	12/18/2009	1/25/2010	22
swe: months 1 6 12 years 2 5 7 10	1/26/2010	1/28/2010	3
swe: months 1 3 6 12 years 2 5 7 10	1/29/2010	10/6/2010	175
swe: months 1 3 6 years 2 5 7 10	10/7/2010	4/11/2011	130
swe: months 1 3 6 12 years 2 5 7 10	4/12/2011	4/13/2011	2
swe: months 1 3 6 years 2 5 7 10	4/14/2011	4/14/2011	1
swe: months 1 3 6 12 years 2 5 7 10	4/15/2011	4/21/2011	5
swe: months 1 3 6 years 2 5 7 10	4/26/2011	12/23/2013	673
swe: months 3 6 years 2 5 7 10	12/27/2013	12/27/2013	1
swe: months 1 3 6 years 2 5 7 10	12/30/2013	8/19/2014	158
swe: months 3 6 years 2 5 7 10	8/20/2014	8/21/2014	2
swe: months 1 3 6 years 2 5 7 10	8/22/2014	10/19/2015	291

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Data Regime	Start Date	End Date	Number of Observations
swe: months 3 6 years 2 5 7 10	10/20/2015	10/20/2015	1
swe: months 1 3 6 years 2 5 7 10	10/21/2015	9/16/2016	228
swe: months 3 6 years 2 5 7 10	9/19/2016	9/19/2016	1
swe: months 1 3 6 years 2 5 7 10	9/20/2016	10/14/2016	19
swe: months 3 6 years 2 5 7 10	10/17/2016	10/17/2016	1
swe: months 1 3 6 years 2 5 7 10	10/18/2016	11/11/2016	19
swe: months 3 6 years 2 5 7 10	11/14/2016	11/14/2016	1
swe: months 1 3 6 years 2 5 7 10	11/15/2016	12/16/2016	24
swe: months 3 6 years 2 5 7 10	12/19/2016	12/19/2016	1
swe: months 1 3 6 years 2 5 7 10	12/20/2016	4/12/2017	80
swe: months 3 6 years 2 5 7 10	4/13/2017	4/18/2017	2
swe: months 1 3 6 years 2 5 7 10	4/19/2017	5/12/2017	17
swe: months 3 6 years 2 5 7 10	5/15/2017	5/15/2017	1
swe: months 1 3 6 years 2 5 7 10	5/16/2017	8/11/2017	61
swe: months 3 6 years 2 5 7 10	8/14/2017	8/14/2017	1
swe: months 1 3 6 years 2 5 7 10	8/15/2017	9/15/2017	24
swe: months 3 6 years 2 5 7 10	9/18/2017	9/18/2017	1
swe: months 1 3 6 years 2 5 7 10	9/19/2017	10/13/2017	19
swe: months 3 6 years 2 5 7 10	10/16/2017	10/17/2017	2
swe: months 1 3 6 years 2 5 7 10	10/18/2017	11/10/2017	18
swe: months 3 6 years 2 5 7 10	11/13/2017	11/14/2017	2
swe: months 1 3 6 years 2 5 7 10	11/15/2017	1/12/2018	40
swe: months 3 6 years 2 5 7 10	1/15/2018	1/16/2018	2
swe: months 1 3 6 years 2 5 7 10	1/17/2018	2/16/2018	23
swe: months 3 6 years 2 5 7 10	2/19/2018	2/19/2018	1
swe: months 1 3 6 years 2 5 7 10	2/20/2018	6/15/2018	79
swe: months 3 6 years 2 5 7 10	6/18/2018	6/19/2018	2
swe: months 1 3 years 2 5 7 10	6/20/2018	6/25/2018	3
swe: months 1 3 6 years 2 5 7 10	6/26/2018	8/10/2018	34
swe: months 3 6 years 2 5 7 10	8/13/2018	8/14/2018	2
swe: months 1 3 years 2 5 7 10	8/15/2018	10/4/2018	37
swe: months 1 3 6 years 2 5 7 10	10/5/2018	10/12/2018	6

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Data Regime	Start Date	End Date	Number of Observations
swe: months 3 6 years 2 5 7 10	10/15/2018	10/16/2018	2
swe: months 1 6 years 2 5 7 10	10/17/2018	10/30/2018	10
swe: months 1 3 6 years 2 5 7 10	10/31/2018	11/5/2018	4
swe: months 3 6 years 2 5 7 10	11/6/2018	11/7/2018	2
swe: months 1 3 6 years 2 5 7 10	11/8/2018	1/11/2019	42
swe: months 3 6 years 2 5 7 10	1/14/2019	1/14/2019	1
swe: months 1 3 6 years 2 5 7 10	1/15/2019	2/19/2019	26
swe: months 3 6 years 2 5 7 10	2/20/2019	3/19/2019	20
swe: months 1 3 6 years 2 5 7 10	3/20/2019	8/30/2021	613
tha: years 1 to 11	9/15/1999	3/8/2000	120
tha: years 1 to 10	3/9/2000	7/5/2000	79
tha: years 1 to 11	7/6/2000	12/29/2000	121
tha: months 1 3 and years 1 to 11	1/3/2001	2/28/2001	40
tha: months 1 3 and years 1 to 13	3/1/2001	4/19/2001	33
tha: months 1 3 and years 1 to 18	4/20/2001	12/28/2001	172
tha: months 1 3 6 and years 1 to 18	1/2/2002	4/12/2002	71
tha: months 1 3 6 and years 1 to 17	4/17/2002	6/28/2002	50
tha: months 1 3 6 and years 1 to 18	7/2/2002	11/7/2002	90
tha: months 1 3 6 and years 1 to 20	11/8/2002	11/13/2002	4
tha: months 1 3 6 and years 1 to 19	11/14/2002	11/13/2003	244
tha: months 1 3 6 and years 1 to 18	11/14/2003	11/12/2004	244
tha: months 1 3 6 and years 1 to 17	11/15/2004	11/11/2005	244
tha: months 1 3 6 and years 1 to 16	11/14/2005	7/6/2006	156
tha: months 1 3 6 and years 1 to 20	7/7/2006	7/12/2006	3
tha: months 1 3 6 and years 1 to 19	7/13/2006	7/12/2007	246
tha: months 1 3 6 and years 1 to 18	7/13/2007	11/29/2007	97
tha: months 1 3 6 and years 1 to 20	11/30/2007	3/18/2008	71
tha: months 1 3 6 and years 1 to 19	3/19/2008	4/24/2008	24
tha: months 1 3 6 and years 1 to 29	4/25/2008	3/20/2009	222
tha: months 1 3 6 and years 1 to 28	3/23/2009	6/23/2009	60
tha: months 1 3 6 and years 1 to 29	6/24/2009	10/13/2009	76
tha: months 1 3 6 and years 1 to 30	10/14/2009	6/30/2010	170

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Data Regime	Start Date	End Date	Number of Observations
tha: months 1 3 6 and years 1 to 29	7/2/2010	10/26/2010	79
tha: months 1 3 6 and years 1 to 30	10/27/2010	3/1/2011	85
tha: months 1 3 6 and years 1 to 50	3/2/2011	6/30/2011	79
tha: months 1 3 6 and years 1 to 49	7/4/2011	6/29/2012	245
tha: months 1 3 6 and years 1 to 48	7/2/2012	6/28/2013	245
tha: months 1 3 6 and years 1 to 47	7/2/2013	6/27/2014	243
tha: months 1 3 6 and years 1 to 46	6/30/2014	6/29/2015	241
tha: months 1 3 6 and years 1 to 45	6/30/2015	11/24/2015	102
tha: months 1 3 6 and years 1 to 50	11/25/2015	6/29/2016	142
tha: months 1 3 6 and years 1 to 49	6/30/2016	6/29/2017	246
tha: months 1 3 6 and years 1 to 48	6/30/2017	11/7/2017	87
tha: months 1 3 6 and years 1 to 49	11/8/2017	6/29/2018	157
tha: months 1 3 6 and years 1 to 48	7/2/2018	1/14/2021	616
Total Observations			**90,720**

Sweden and Thailand account for the very large number of data regimes in the 13-country data set. We discuss the implications of this for modeling strategy below.

Because our Heath, Jarrow and Morton analysis makes use of a yield curve with 91-day (quarterly) forward rate segments, the next step in data model validation is to fit 91-day forward rates to the raw coupon-bearing bond yields. The smoothness of the resulting forward rates will be a function of both the quality of the raw data from a smoothness point of view and the smoothness implied by the secondary smoothing process.   To ensure the maximum smoothness from the secondary smoothing process, we use the maximum smoothness forward rate methodology of Adams and van Deventer [1994], as corrected in van Deventer and Imai [1996]. Adams and van Deventer show that the maximum smoothness method overcomes the problems of the cubic spline approach of McCulloch, and, unlike the Svensson [1994] approach, allows for a perfect fit to the raw data provided by third-party data providers, such as the U.S. Department of the Treasury. See Jarrow [2014] for information on the problems with Svensson yield curve fitting.

We then conduct a visual inspection of the resulting forward rates implied by the raw data. The smoothness of the quarterly forward rate curve can also be measured quantitatively. For a yield curve that consists of N quarterly forward rates, the discrete smoothness statistic at time t, $Z_N(t)$, is the sum of the squared second differences in the forward rates, as explained by Adams and van Deventer [1994]. A closed form continuous smoothing statistic can also be calculated when the functional form of the continuous forward rate is known (as it is in this case). The discrete smoothness statistic is given here:

$$Z_N(t) = \sum_{i=3}^{N} \left[\left(f_i(t) - f_{i-1}(t) \right) - \left(f_{i-1}(t) - f_{i-2}(t) \right) \right]^2$$

A wide variety of smoothing techniques were used to create the "on the run" par coupon bond yields in the 13-country data set.   An analysis of the maximum smoothness forward rates implied by older data in the Japan data set and most data in the Thai data set indicated that an unconventional smoothing process had been used in the very detailed set of yields provided by the Ministry of Finance in Japan and by the Thai Bond Market Association. Rather than discard these two very interesting data sets, we only used the maturities at which bonds were actually issued by the governments of Japan and Thailand.

- We then created smoothed yield curves using Kamakura Risk Manager version 10 for each observation data in each country.
- We calculated the discrete smoothness statistics for 10 years and 30-years.
- We dropped any observation in which the smoothness statistic was higher (less smooth) than any of the more than 14,000 observations in the U.S. Treasury set. June 21, 1982 was the least smooth yield curve in the U.S. history.

74 observations from these countries were dropped:

- Australia, 4 observations
- Spain, 42 observations

- Sweden, 24 observations
- United Kingdom, 4 observations

The 4 observations dropped in Australia were clearly due to errors in data storage by the third-party data source used (it was not the Reserve Bank of Australia).

We applied a final screen for outliers in the process of deriving the stochastic volatility functions. Yield curves on the following days were omitted from estimation because they implied implausible forward rate shocks in the first 91-day forward rate, which provides the basis for the first factor in the model. There were additional dropped observations associated with other factors in the term structure model as well.

- Australia, July 18, 1997
- Australia, July 30, 1997
- Australia, February 18, 1998
- Russia, December 16, 2014
- Spain, February 10, 2009
- Spain, February 11, 2009
- Spain, February 16, 2009
- Spain, February 26, 2009
- Spain, April 27, 2009
- Spain, May 6, 2009
- Spain, May 12, 2009
- Spain, May 13, 2009
- Spain, May 18, 2009
- Spain, May 28, 2009
- Spain, July 7, 2009
- Spain, July 27, 2009
- Spain, October 6, 2009
- Spain, March 1, 2010
- Spain, March 31, 2010
- Spain, April 15, 2010
- Spain, May 12, 2010
- Spain, May 31, 2010
- Spain, June 1, 2010
- Spain, June 4, 2010
- Spain, June 8, 2010
- Spain, June 9, 2010
- Spain, June 17, 2010
- Spain, June 24, 2010
- Spain, June 25, 2010
- Spain, June 29, 2010
- Spain, June 30, 2010
- Spain, July 6, 2010
- Spain, July 7, 2010
- Spain, July 9, 2010
- Spain, July 13, 2010
- Spain, July 15, 2010

- Spain, July 29, 2010
- Spain, September 6, 2010
- Spain, October 28, 2010
- Spain, December 10, 2010
- Spain, December 14, 2010
- Spain, August 29, 2011
- Spain, March 28, 2012
- Spain, June 1, 2012
- Spain, June 11, 2012
- Spain, June 12, 2012
- Spain, June 13, 2012
- Spain, June 14, 2012
- Spain, June 15, 2012
- Spain, June 18, 2012
- Spain, June 20, 2012
- Spain, June 22, 2012
- Spain, June 27, 2012
- Spain, July 5, 2012
- Spain, August 31, 2012
- Spain, September 13, 2012
- Spain, January 30, 2013
- Spain, February 28, 2013
- Spain, March 7, 2013
- Spain, April 11, 2013
- Spain, April 15, 2013
- Spain, June 6, 2013
- Spain, July 11, 2013
- Spain, July 15, 2013
- Spain, December 24, 2015
- Spain, June 19, 2017
- Spain, August 7, 2017
- Spain, August 30, 2017
- Spain, September 18, 2017
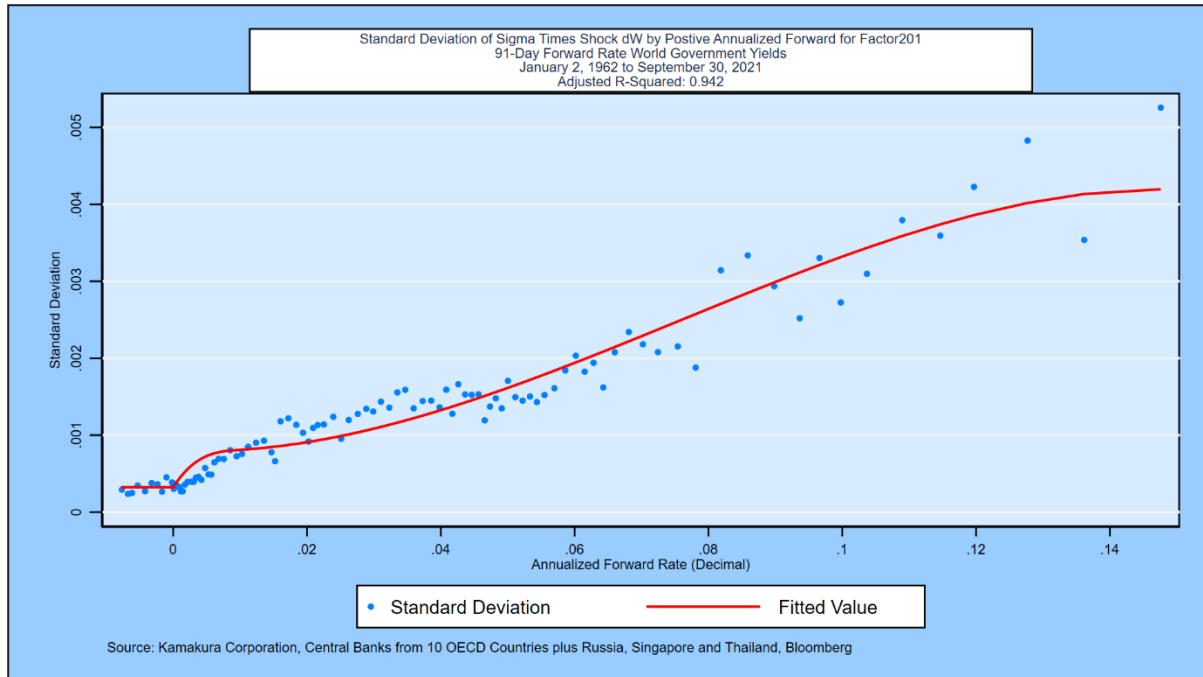- Sweden, December 18, 2018
- Sweden, December 19, 2018

II. Constant versus Stochastic Volatility

Constant volatility ("affine") term structure models are commonly used for their ease of simulation and estimation of "future expected rates" in order to determine the "term premium" in current yields. Prominent examples are Adrian, Crump and Moench [2013], Kim and Wright [2005], and Duffie and Kan [1996]. On the other hand, the weight of the empirical evidence in most of the countries studied to date indicates that interest rate volatility does vary by the level of the corresponding forward rate. To illustrate that fact, we studied the shortest forward rate among the 13-country yield curves on a daily basis from January 2, 1962 through September 30, 2021. We ordered the data from lowest forward rate level to highest forward rate level. We

formed non-overlapping groups using the larger of 50 observations or $1/100^{th}$ of total observations each and calculated both the standard deviation of 91-day forward rate changes and the mean beginning-of-period forward rate in each group. The results are plotted in Exhibit I:

Exhibit I

Standard Deviation of Sigma Times Shock dW by Postive Annualized Forward for Factor201
91-Day Forward Rate World Government Yields
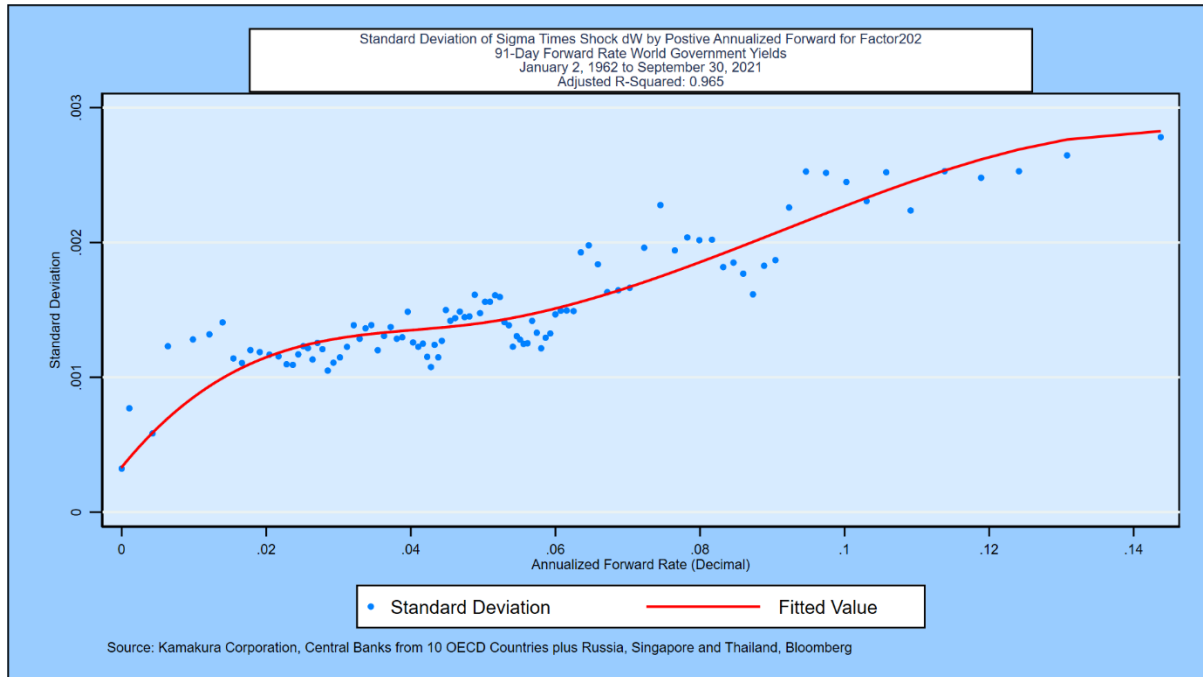January 2, 1962 to September 30, 2021
Adjusted R-Squared: 0.942

A 2-segment cubic spline of annualized forward rates explains 94.2%[2] of the variation in the standard deviation of forward rate changes for these ordered groups. This is the volatility function used when extracting the first random factor driving the 13-country yield curves. Note that the right-hand side of the curve has been constrained to have a first derivative of zero at a high level of rates.[3] The rise in volatility in higher rate environments has been confirmed in the government securities markets for each of the 13-country stand-alone data sets. Exhibit II shows the results for the second risk factor in the 13-country data set, the idiosyncratic movements in the 91-day forward rate maturing in 10 years:

---

[2] Note that the simple r-squared is affected by the heavy weighting on the 0 maturity point to ensure volatility at negative and zero rate levels is consistent with international experience.

[3] This constraint is one method for imposing the cap in stochastic volatilities suggested by Heath, Jarrow, and Morton [*Econometrica*, 1992] to prevent a positive possibility of (a) infinitely high rates or (more practically) (b) unrealistically high rates.

Exhibit II

Standard Deviation of Sigma Times Shock dW by Positive Annualized Forward for Factor202
91-Day Forward Rate World Government Yields
January 2, 1962 to September 30, 2021
Adjusted R-Squared: 0.965

Source: Kamakura Corporation, Central Banks from 10 OECD Countries plus Russia, Singapore and Thailand, Bloomberg

The cubic spline stochastic volatility specification explains 96.5% of the observed variation in forward rate volatility in the quarterly forward rate maturing at the 10-year point on the 13-country yield curves. We have imposed the same constraint on the first derivative and require that the fitted volatility not be less than the lowest observed volatility, which we discuss later in this section.

In Appendix C, we compare the span of the 13-country data set with the range of rates and maturities contained in certain of the 13-country stand-alone data sets. The U.S. data set spans 88.08% of the World experience, the highest amount among the 13 countries studied.

We use three statistical tests to determine whether or not the hypothesis of normality for forward rates and zero-coupon bond yields should be rejected at the 5% level: the Shapiro-Wilk test, the Shapiro-Francia test, and the skew test, all of which are available in common statistical packages. The 13-country history is now much larger than the maximum number of observations allowed by these tests. In early work, when the sample size was smaller, the hypothesis of normality was rejected at each 91-day forward rate maturity date for all 13 countries. We re-examine this hypothesis as needed using smaller samples from the full data set on a regular basis.

The rejection of the hypothesis of normality is problematic for constant coefficient or "affine" term structure models. In most of the other countries studied, the hypothesis of normality has been rejected strongly as well. Given these results, we proceed with caution on the implementation of the affine model.

In Chapter 3 of Advanced Financial Management (second edition, 2013), van Deventer, Imai and Mesler analyze the frequency with which U.S. Treasury forward rates move up together, down together or remain unchanged. This exercise informs the Heath, Jarrow, and Morton parameter fitting process and is helpful for the model validation questions posed in the Appendix. We perform the yield curve shift analysis using 90,720 days of quarterly forward rates for the 13-country consolidated yield curves. We analyze the daily shifts in the forward rates on each business day from January 2, 1962 through September 30, 2021. The results are given in Table II:

Table II

Kamakura Corporation		
HJM 12 Factor Model		
World Government		
Using Daily Data from January 2, 1962 through September 30, 2021		
Date of Analysis: September 30, 2021		
Type of Yield Shift	Number of Observations	Percent of Observations
All yields shift up	3,762	4.15
All yields shift down	9,323	10.28
All yields are unchanged	328	0.36
Yield curve twists	77,307	85.21
Total	90,720	100.00

Kamakura Corporation, Central Banks from 10 OECD Countries plus Russia, Singapore and Thailand, Bloomberg

Yield curve shifts were all positive, all negative, or all zero 4.15%, 10.28%, and 0.36% of the time, a total of 14.79% of all business days. The predominant yield curve shift was a twist, with a mix of positive changes, negative changes, or zero changes. These figures are similar to those for the 13-country stand-alone data sets as well. These twists, which happen 85.21% of the time in the consolidated data base, cannot be modeled accurately with the conventional implementation of one-factor term structure models.

Another important aspect of yield curves is the number of local minima and maxima that have occurred over the modeling period. The results for the consolidated data set are given in Table III:

Table III

Analysis of Number of Local Minima and Maxima Each Day

Number of Humps	Number of Observations	Percent of Observations
0 local minimum and maximum	28,800	31.75
1	23,439	25.84
2	18,978	20.92
3	9,564	10.54
4	4,265	4.70
5	2,138	2.36
6	2,037	2.25
7	470	0.52
8	818	0.90
9	38	0.04
10 or more	173	0.19

Kamakura Corporation, Central Banks from 10 OECD Countries plus Russia, Singapore and Thailand, Bloomberg

The number of days with 0 or 1 humps (defined as the sum of local minima and maxima on that day's yield curve) was 57.59% of the total observations in the data set.

Finally, before proceeding, we count the number of occurrences of negative rates for each forward rate segment of the yield curve over the history provided by the consolidated data base.

Table IV

Kamakura Corporation
HJM 12 Factor Model
World Government
Using Daily Data from January 2, 1962 through September 30, 2021
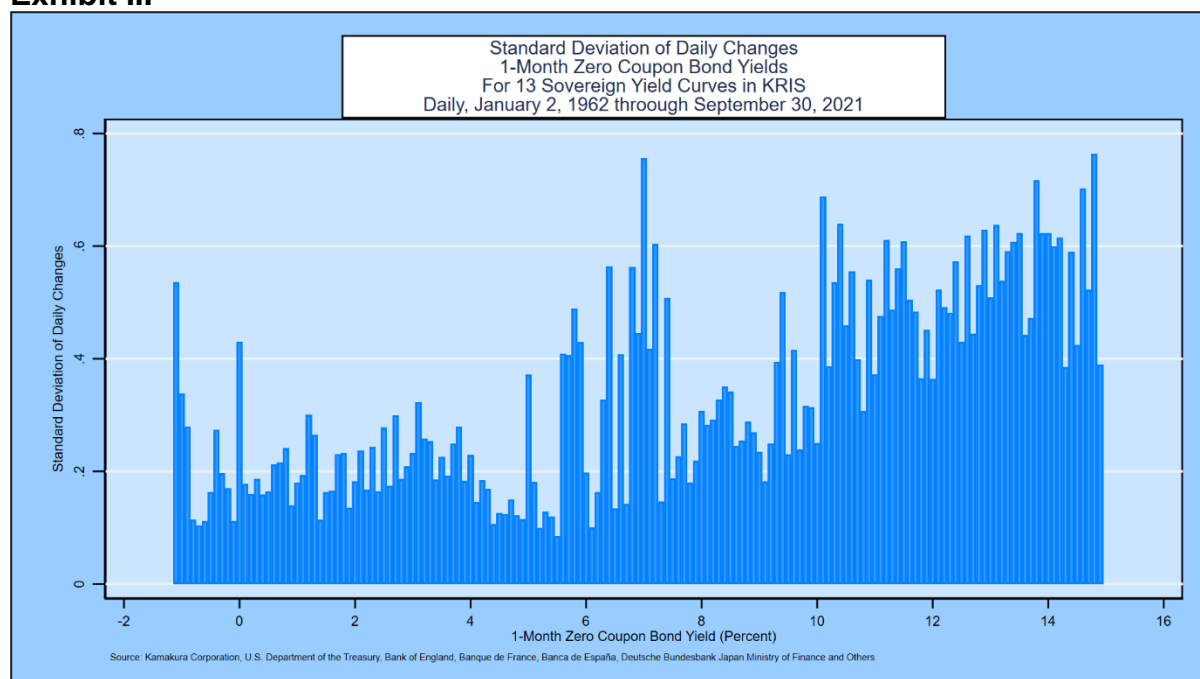Date of Analysis: September 30, 2021
Source: Kamakura Corporation, Central Banks from 10 OECD Countries plus Russia, Singapore and Thailand, Bloomberg

| Count of Negative 91-Day Forward Rate Forward Rates | | | | | Standard Deviation of Change in Forwards | | |
Period Number	Observations	Negative	Zero	Positive	Negative	Zero	Positive
1	90,720	9,426	0	81,294	0.0438%		0.1762%
2	90,720	9,158	0	81,562	0.0454%		0.1752%
3	90,720	9,387	1	81,332	0.0463%	0.0044%	0.1805%
4	90,720	9,435	34	81,251	0.0476%		0.1798%
5	90,720	8,969	1	81,750	0.0485%		0.1830%
6	90,720	8,243	4	82,473	0.0590%	0.0214%	0.1828%
7	90,720	7,626	3	83,091	0.0677%		0.1814%
8	90,720	6,849	0	83,871	0.0720%		0.1798%
9	90,720	6,340	0	84,380	0.0719%		0.1780%
10	90,720	5,786	0	84,934	0.0676%		0.1760%
11	90,720	5,201	0	85,519	0.0620%		0.1734%
12	90,720	4,711	0	86,009	0.0570%		0.1706%

Table IV confirms that 91-day forward rates have been negative for approximately 5-10% of total observations for each of the first 12 forward rate segments. Rates have been negative for 12.044% of observations for 1-month yields in 13 countries over the same period, as shown in Appendix A.

Observed volatility in 1-month zero-coupon bond yields as a function of interest rate level for the standard 13-country World data set is shown in Exhibit III. Volatility trends upward as the yields at the beginning of the measurement period rise:

Exhibit III

Standard Deviation of Daily Changes
1-Month Zero Coupon Bond Yields
For 13 Sovereign Yield Curves in KRIS
Daily, January 2, 1962 throough September 30, 2021

Source: Kamakura Corporation, U.S. Department of the Treasury, Bank of England, Banque de France, Banca de España, Deutsche Bundesbank Japan Ministry of Finance and Others

III. Fitting Heath, Jarrow, and Morton Parameters

A simple first step in constructing a multi-factor Heath, Jarrow and Morton model is to conduct principal components analysis on the forward rates that make up the relevant yield curve. For the 13-country data set, at its longest maturity[4], these quarterly segments consist of one three-month spot rate and 119 forward rates. Over 33,317 observations, the principal components analysis indicates in Table V that the first factor explains only 61.97% of the movement in forward rates over the full curve. For a high degree of explanatory power, the principal components analysis indicates that 12 to 17 factors will be necessary.

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[4] There is an increasing amount of data for maturities of longer than 30 years, but that long-term data spans only a narrow range of rates and, therefore, was not included in this study.

Table V

```
Principal components/correlation                Number of obs    =    33,317
                                                Number of comp.  =        25
                                                Trace            =       119
         Rotation: (unrotated = principal)      Rho              =    1.0000
```

Component	Eigenvalue	Difference	Proportion	Cumulative
Comp1	73.7407	50.1597	0.6197	0.6197
Comp2	23.5809	13.5004	0.1982	0.8178
Comp3	10.0806	5.18331	0.0847	0.9025
Comp4	4.89725	1.71649	0.0412	0.9437
Comp5	3.18075	1.76765	0.0267	0.9704
Comp6	1.41311	.600689	0.0119	0.9823
Comp7	.812417	.326594	0.0068	0.9891
Comp8	.485824	.210014	0.0041	0.9932
Comp9	.27581	.084297	0.0023	0.9955
Comp10	.191513	.0644983	0.0016	0.9971
Comp11	.127015	.0430437	0.0011	0.9982
Comp12	.0839712	.0180092	0.0007	0.9989
Comp13	.0659619	.0349407	0.0006	0.9995
Comp14	.0310212	.0173275	0.0003	0.9997
Comp15	.0136937	.00584202	0.0001	0.9998
Comp16	.00785169	.00125713	0.0001	0.9999
Comp17	.00659456	.00381958	0.0001	1.0000
Comp18	.00277498	.00130827	0.0000	1.0000
Comp19	.00146671	.00107567	0.0000	1.0000
Comp20	.000391042	.000116255	0.0000	1.0000
Comp21	.000274787	.00020228	0.0000	1.0000
Comp22	.0000725073	.0000474844	0.0000	1.0000
Comp23	.0000250229	8.80490e-06	0.0000	1.0000
Comp24	.000016218	4.30419e-06	0.0000	1.0000
Comp25	.0000119138	5.01846e-06	0.0000	1.0000
Comp26	6.89534e-06	5.61388e-06	0.0000	1.0000
Comp27	1.28146e-06	1.28146e-06	0.0000	1.0000
Comp28	0	0	0.0000	1.0000
Comp29	0	0	0.0000	1.0000
Comp30	0	0	0.0000	1.0000

With this analysis as background, we begin the Heath, Jarrow, and Morton fitting process.

In the studies done so far, the number of statistically significant factors are summarized below:

- Australia: Commonwealth Government Securities, 14 factors
- Canada: Government of Canada Securities, 7 factors
- France: French Government Bonds, 7 factors
- Germany: Bunds, 15 factors
- Italy: Italian Government Bonds, 11 factors
- Japan: Japanese Government Bonds, 8 factors
- Russia: Russian Government Bonds, 11 factors

- Singapore: Singapore Government Securities, 9 factors
- Spain: Spanish Government Securities, 11 factors
- Sweden: Swedish Government Securities, 11 factors
- Thailand   Thai Government Securities, 11 factors
- United Kingdom: Government Securities, 14 factors
- United States: Treasury Securities, prior version, 10 factors
- World: 13 countries' government securities, 12 factors

We now fit a multi-factor Heath, Jarrow, and Morton model to the 13-country zero-coupon yield data from January 2, 1962 to September 30, 2021. For computational simplicity, we compress the data regimes numbered in the right-hand column of Table I to three regimes. The first is for observations where no maturity longer than 10 years was reported. The second is for those observations where no maturity longer than 20 years was reported. The third regime extends to the full 30-year maximum in the data set.

The procedures used to derive the parameters of a Heath, Jarrow and Morton model are described in detail in Jarrow and van Deventer (June 16, 2015 and May 5, 2017). We followed these steps to estimate the parameters of the model:
- We extract the zero-coupon yields and zero-coupon bond prices for all quarterly maturities out to 30 years for all daily observations for which the 30-year zero-coupon yield is available. For other observations, we extended the analysis to the longest maturity available, which varies by data regime. This is done using Kamakura Risk Manager, version 10.1, using the maximum smoothness forward rate approach to fill the quarterly maturity gaps in the zero-coupon bond data.
- We use overlapping 91-day intervals to measure changes in forward rates, avoiding the use of "quarterly" data because of the unequal lengths of calendar quarters. Because overlapping observations trigger auto-correlation, "HAC" (heteroscedasticity and autocorrelation consistent) standard errors are used. The methodology is that of Newey-West with 91-day lags.
- We consider twelve potential explanatory factors: the idiosyncratic portion of the movements in 91-day forward rates that mature in 6 months, 1 year, 1.5, 2, 3, 5, 7, 10, 15, 20, 25, and 30 years.
- We calculate the discrete changes in forward returns as described in the parameter technical guide. Because the discrete changes are non-linear in the no-arbitrage framework of Heath, Jarrow, and Morton, we use generalized linear models and the proper HJM functional form to fit interest rate volatility.
- We use a different non-linear regression for each segment of the yield curve. We considered a panel-based approach, but we rejected it for two reasons: first, the movement of parameters as maturity lengthens is complex and not easily predictable before estimation; second, the residual unexplained error in forward rates is very, very small, so the incremental merit of the panel approach is minimal.
- We then begin the process of creating the orthogonalized risk factors that drive interest rates using the Gram-Schmidt procedure. These factors are assumed, without loss of generality, to be uncorrelated independent random

variables that have a normal distribution with mean zero and standard deviation of 1.

- Because interest volatility is assumed to be stochastic, simulated out-of-sample forward rates will not in general be normally distributed. We also calculate constant volatility parameters and choose the most accurate from the constant volatility and stochastic volatility models estimated.
- In the estimation process, we added factors to the model as long as each new factor provided incremental explanatory power. The standard suite of models in both cases includes 1 factor, 2 factors, 3 factors, 6 factors and "all factors," which varies by country.

We postulate that unannualized interest rate volatility for each forward rate maturity k is a cubic function of the annualized forward rate that prevails for the relevant risk factor j at the beginning of each 91-day period:

$$\sigma_{jk} = \max\left[b_{0,jk}, b_{0,jk} + b_{1,jk}f + b_{2,jk}f^2 + b_{3,jk}f^3\right] \, if \, f > 0,$$
$$\sigma_{jk} = b_{0,jk} \, if \, f \leq 0,$$

When the initial forward rate is negative, we postulate that interest rate volatility is a constant[5]. Using Japan volatility data reported above, the measured $b_{0,jk}$ was 0.1408%. Using the 13-country World model, the value of $b_{0,jk}$ is 0.0478%.

We use the resulting parameters and accuracy tests to address the hypothesis that a one-factor model is "good enough" for modeling 13-country government yields in Appendix A. We report the accuracy results for 1, 2, 3, 6 and all (12) factors. The factors are the idiosyncratic variation in 91-day forward rates at each of 12 maturities. The factors, described by the maturity of the quarterly forward rate used, are added to the model in this order:

- **Data Regime 1 (Maturities of 10 years or less)**

  - Factor 1:   6 months
  - Factor 2:   10 years
  - Factor 3:   3 years
  - Factor 4:   7 years
  - Factor 5:   1 year
  - Factor 6:   5 years
  - Factor 7:   2 years
  - Factor 8:   1.5 years

- **Data Regime 2 (Maturities longer than 10 years up to 20 years)**

  - Factor 9:   15 years
  - Factor 10:   20 years

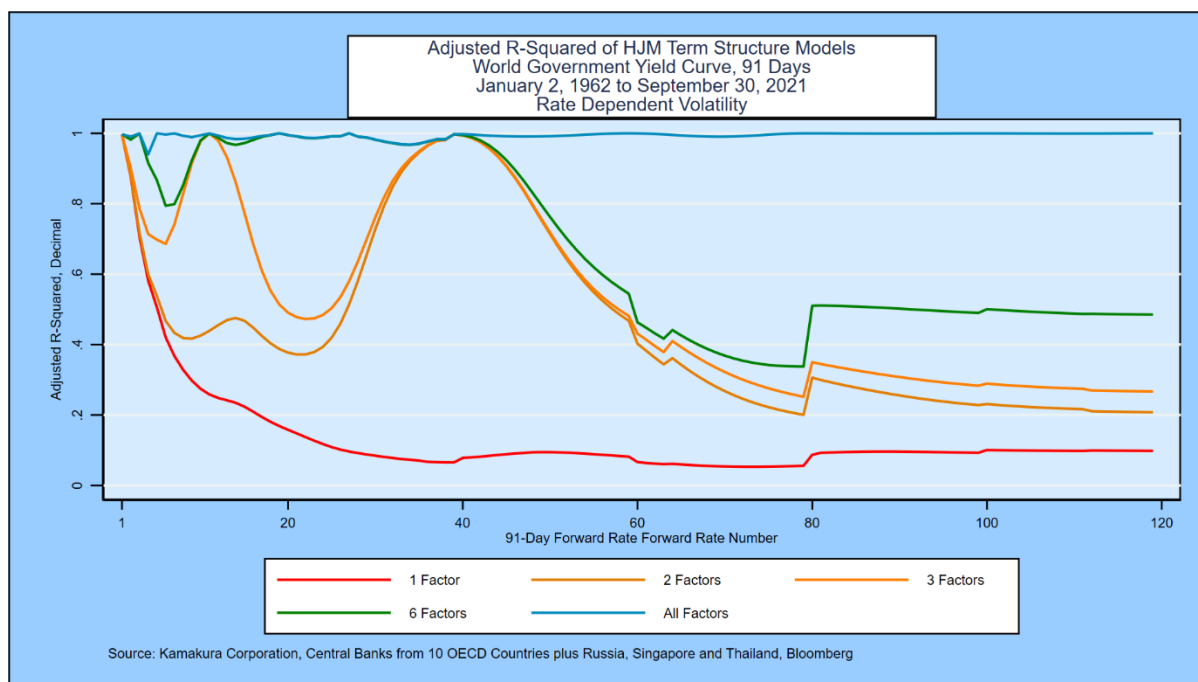- **Data Regime 3 (Maturities longer than 20 years up to 30 years)**

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[5] We make this assumption while waiting for more data on negative rates to accumulate. At the appropriate time, a more interesting functional form will be employed.

- o Factor 11:   25 years
- o Factor 12:   30 years

Exhibit IV summarizes the adjusted r-squared for the non-linear equations for each of the 119 quarterly forward rate segments that make up the 13-country yield curves:
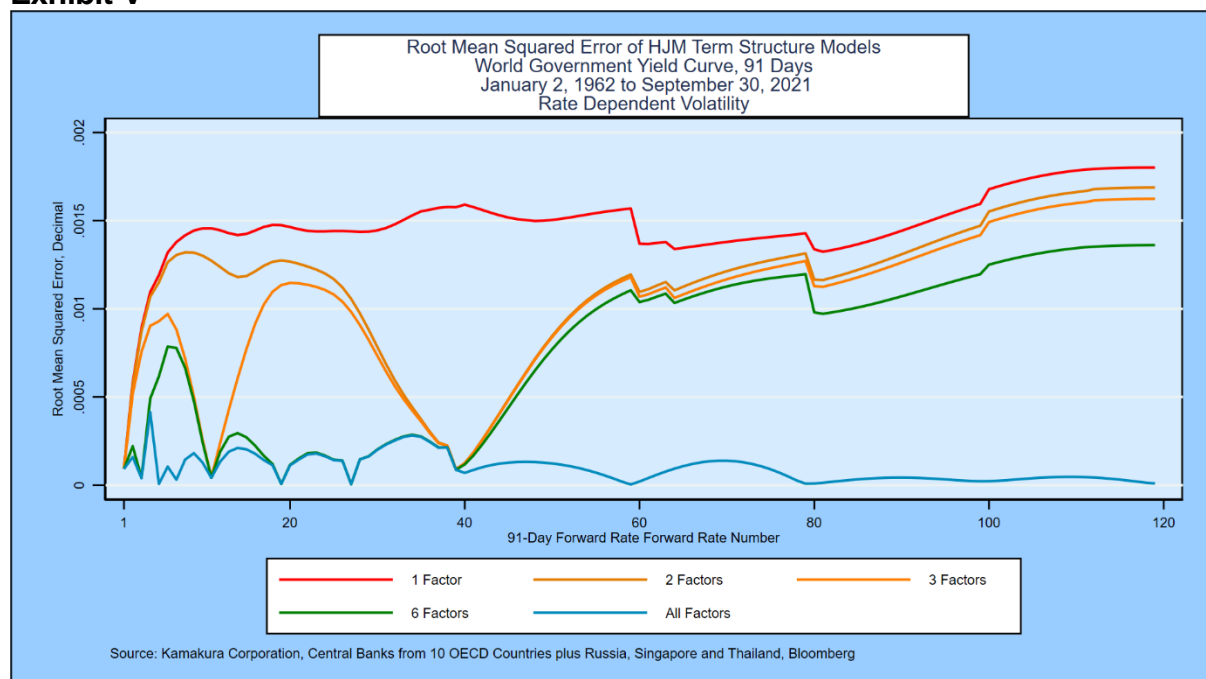
Exhibit IV

Adjusted R-Squared of HJM Term Structure Models
World Government Yield Curve, 91 Days
January 2, 1962 to September 30, 2021
Rate Dependent Volatility

The adjusted r-squared for the best practice model over each of the forward rates is plotted in blue and is near 100% for all 119 quarterly segments of the yield curve. The one-factor model in red, by contrast, does a poor job of fitting 91-day movements in the quarterly forward rates. The adjusted r-squared is good, of course, for the first forward rate since the short rate is the standard risk factor in a one-factor term structure model. Beyond the first quarter, however, explanatory power declines rapidly. The adjusted r-squared of the one-factor model never exceeds 20% after the first 17 quarterly forward rates and is below that level at most maturities.

The root mean squared error for the 1, 2, 3, 6 and all (12) factor stochastic volatility model is shown in Exhibit V.

Exhibit V

Root Mean Squared Error of HJM Term Structure Models
World Government Yield Curve, 91 Days
January 2, 1962 to September 30, 2021
Rate Dependent Volatility

Source: Kamakura Corporation, Central Banks from 10 OECD Countries plus Russia, Singapore and Thailand, Bloomberg

The root mean squared error for the 12-factor model is less than 0.05% at every maturity along the yield curve. This result should not come as a surprise to a serious analyst, because it is very similar to the results of the best practice Heath, Jarrow, and Morton term structure models for each the 13 government bond markets studied on a stand-alone basis.

Bayesian Considerations in Model Validation

Kamakura term structure model validation is conducted in the spirit of Bayesian iterative model building as outlined by Gelman et al. This quote[6] from Gelman et al [2013] explains the Bayesian estimation process:

"The process of Bayesian data analysis can be idealized by dividing it into the following three steps:
1. Setting up a full probability model—a joint probability distribution for all observable and unobservable quantities in a problem. The model should be consistent with knowledge about the underlying scientific problem and the data collection process.
2. Conditioning on the observed data: calculating and interpreting the appropriate posterior distribution—the conditional probability distribution of the unobserved quantities of ultimate interest, given the observed data.
3. Evaluating the fit of the model and the implications of the resulting posterior distribution: how well does the model fit the data, are the substantive conclusions reasonable, and how sensitive are the results to the modeling assumptions in step 1? In response, one can alter or expand the model and repeat the three steps."

---

[6] Gelman et all [2013], page 3.

Jarrow and van Deventer go on to explain that the iterative process described above by Gelman *et al* is especially important in fitting Heath, Jarrow, and Morton parameters for the following reasons:

a. Negative interest rates have been observed in Japan, Hong Kong, and many European countries, but many other countries, including the U.S., have yet to experience negative rates. In the U.S. case, the U.S. Department of the Treasury notes on its website that it overrides observed negative yields in the market with zero values.

b. The "knowledge about the underlying scientific problem" from the historical data available is as follows: (1) negative rates are possible, (2) they are much less likely to occur than positive rates, (3) interest rate volatility that results when rates are negative is of high interest but the historical data is either limited or non-existent, depending on the country, and (4) an international data set would best shed light on this and other HJM issues.

There are other issues relevant to estimation:

c. As noted by Heath, Jarrow, and Morton [1992], stochastic volatility driven by interest rate levels must be capped to avoid a positive probability of infinitely high interest rates.

d. Subject to this cap, most market participants expect interest rate volatility to rise as rates rise and that the interest rate volatility that prevails when rates are negative represents the lowest level of volatility that would prevail. Historical experience with negative rates so far around the world makes it clear that interest rate volatility does not go to zero at any rate level.

e. Most market participants believe that the empirical drift in forward rates that occurs (i.e., the change in observed empirical interest rates in the case where all interest rate shocks are zero) varies by the level of interest rates. The stochastic volatility model described in this paper assumes that empirical drift is a cubic function of annualized forward rates.

To summarize, a model validation effort in the Bayesian spirit would address at least these issues:

• Tests of smoothness of simulated curves
• Tests to confirm existence of negative rates in selected circumstances in the simulation
• Comparison of simulated risk neutral and empirical yields
• Time series distribution of simulated risk neutral and empirical yields
• Tests for the reasonableness of empirical drift in rates at each segment of the yield curve and stress tests of the rate factor volatility functions for reasonableness over the full range of rates that have been observed in the past. These tests are shown in Appendix B

We conduct an inspection of these issues with the aid of a forward-looking out-of-sample simulation of U.S. Treasury yields from a starting point of October 15, 2021 using both the World term structure model coefficients and the U.S. single-country model coefficients with the following specifications:

- Yield curve: U.S. Treasury yields
- Date of yields: October 15, 2021
- Number of scenarios: 500,000
- Simulation time horizon: 30 years
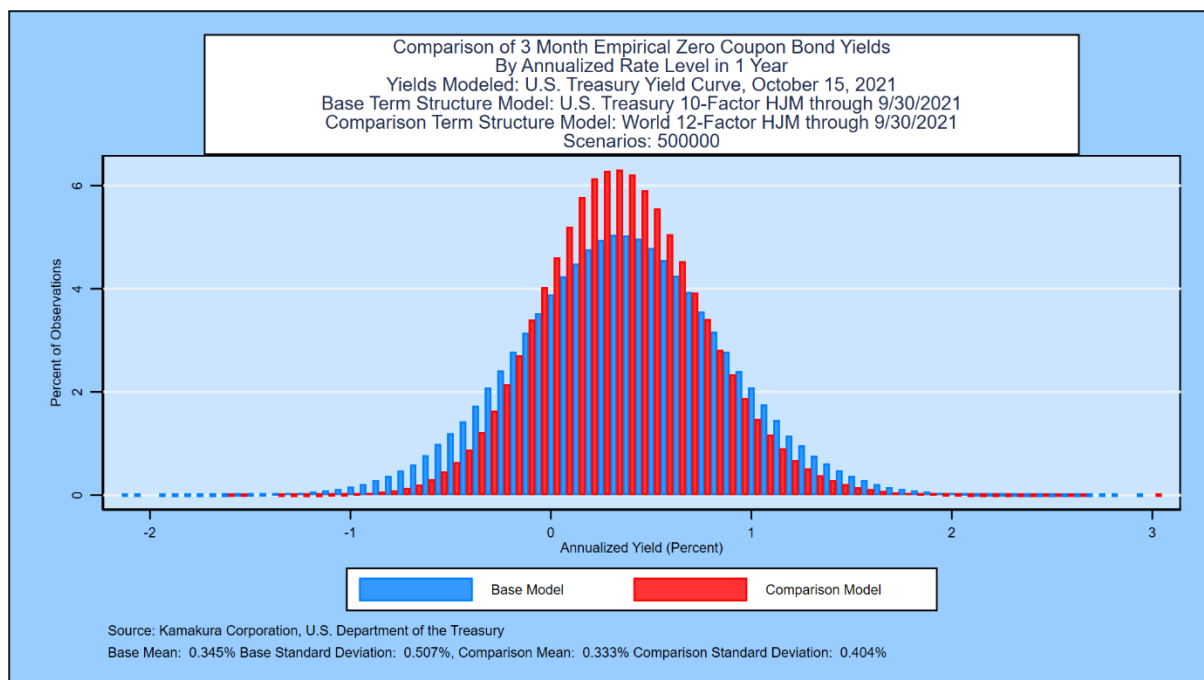- Simulation periodicity: 91 days (quarterly)

Smoothness Validation

First, we select a random sample of 10 scenarios at each time step and visually examine them for smoothness. We can also use the discrete formula for smoothness given above to identify any outliers and examine the scenarios in question. These graphs provide informal confirmation that nothing in the model estimation procedure has introduced artificial lumpiness in the simulated yield curves. A quantitative assessment of the smoothness of all 500,000 yield curves at each time step would provide the more formal confirmation that the yield curves simulated are realistically smooth. For samples of the graphics used, please contact the author via info@kamakuraco.com.

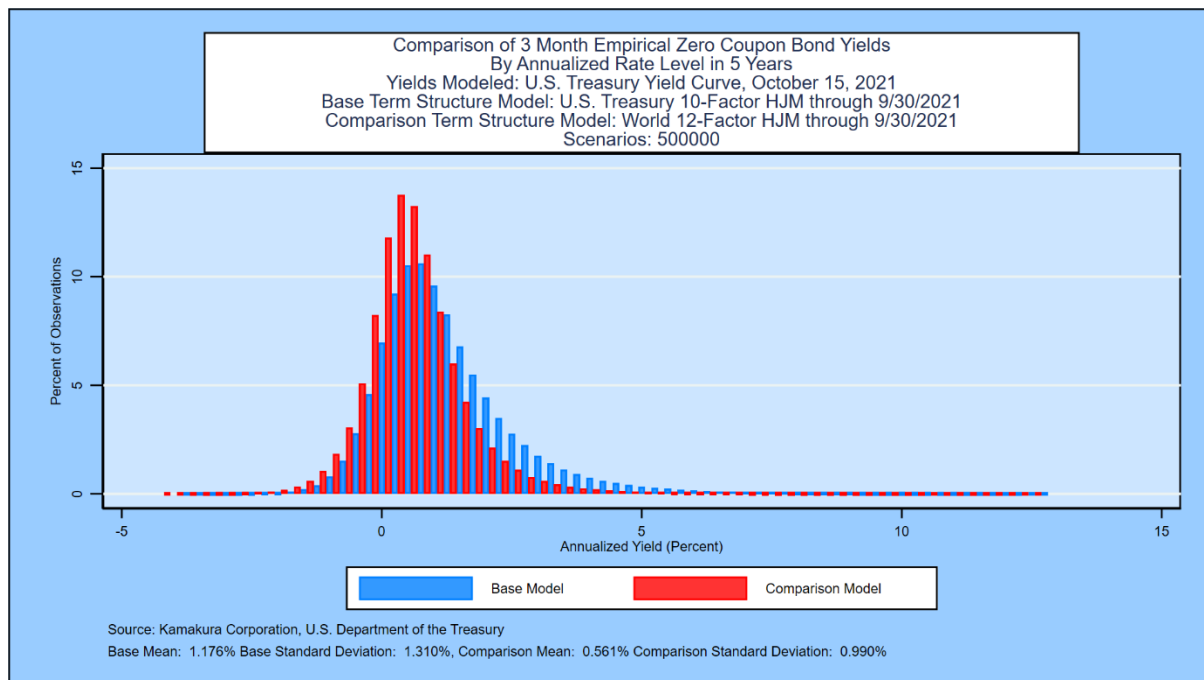**Distribution of Simulated Risk Neutral and Empirical Rates**

We now examine the probability distributions of empirical simulated rates at various maturities over time. We seek to detect visually any points in time at which the simulated distribution of yields is strange or unrealistic. In the following exhibits, we focus on three-month U.S. Treasury yields for both the World and U.S.-only models:

Exhibit VI: Three-Month U.S. Treasury Yields at 1 Year

Comparison of 3 Month Empirical Zero Coupon Bond Yields
By Annualized Rate Level in 1 Year
Yields Modeled: U.S. Treasury Yield Curve, October 15, 2021
Base Term Structure Model: U.S. Treasury 10-Factor HJM through 9/30/2021
Comparison Term Structure Model: World 12-Factor HJM through 9/30/2021
Scenarios: 500000

Source: Kamakura Corporation, U.S. Department of the Treasury
Base Mean: 0.345% Base Standard Deviation: 0.507%, Comparison Mean: 0.333% Comparison Standard Deviation: 0.404%

The U.S. only model is denoted the base model and shown in blue. The World or "comparison" model, is shown in red. The World model at a 1-year horizon shows less volatility in simulated empirical 3-month Treasury bill yields.
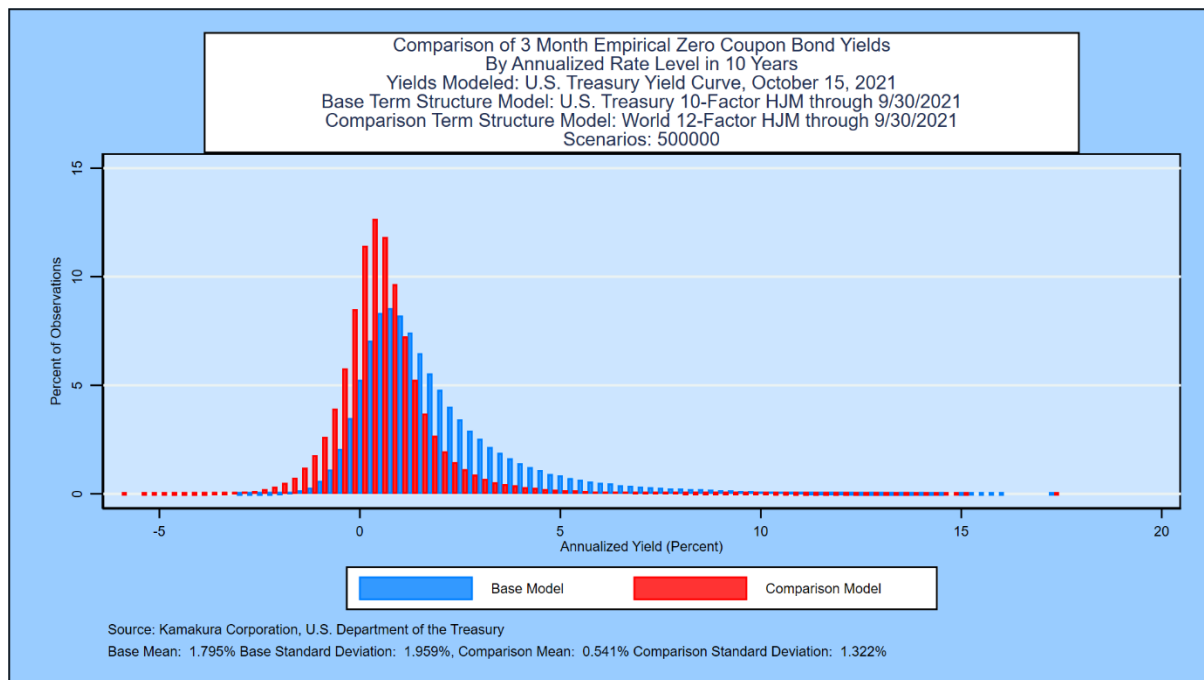
Exhibit VII: Three-Month U.S. Treasury Yields at 5 Years

Comparison of 3 Month Empirical Zero Coupon Bond Yields
By Annualized Rate Level in 5 Years
Yields Modeled: U.S. Treasury Yield Curve, October 15, 2021
Base Term Structure Model: U.S. Treasury 10-Factor HJM through 9/30/2021
Comparison Term Structure Model: World 12-Factor HJM through 9/30/2021
Scenarios: 500000

Source: Kamakura Corporation, U.S. Department of the Treasury
Base Mean: 1.176% Base Standard Deviation: 1.310%, Comparison Mean: 0.561% Comparison Standard Deviation: 0.990%

By the 5-year point in the simulation, the red bars of the World model show a higher level of negative observations and are visibly skewed to the left compared to the U.S.-only base model. This pattern continues through the 10-year and 15-year time horizons.

Exhibit VIII: Three-Month U.S. Treasury Yields at 10 Years

Comparison of 3 Month Empirical Zero Coupon Bond Yields
By Annualized Rate Level in 10 Years
Yields Modeled: U.S. Treasury Yield Curve, October 15, 2021
Base Term Structure Model: U.S. Treasury 10-Factor HJM through 9/30/2021
Comparison Term Structure Model: World 12-Factor HJM through 9/30/2021
Scenarios: 500000

Source: Kamakura Corporation, U.S. Department of the Treasury
Base Mean: 1.795% Base Standard Deviation: 1.959%, Comparison Mean: 0.541% Comparison Standard Deviation: 1.322%

At the 10-year point in the simulation, World model empirical 3-month yields continue to show more negative observations and lower volatility than the U.S.-only model.

Exhibit IX: Three-Month U.S. Treasury Yields at 15 Years
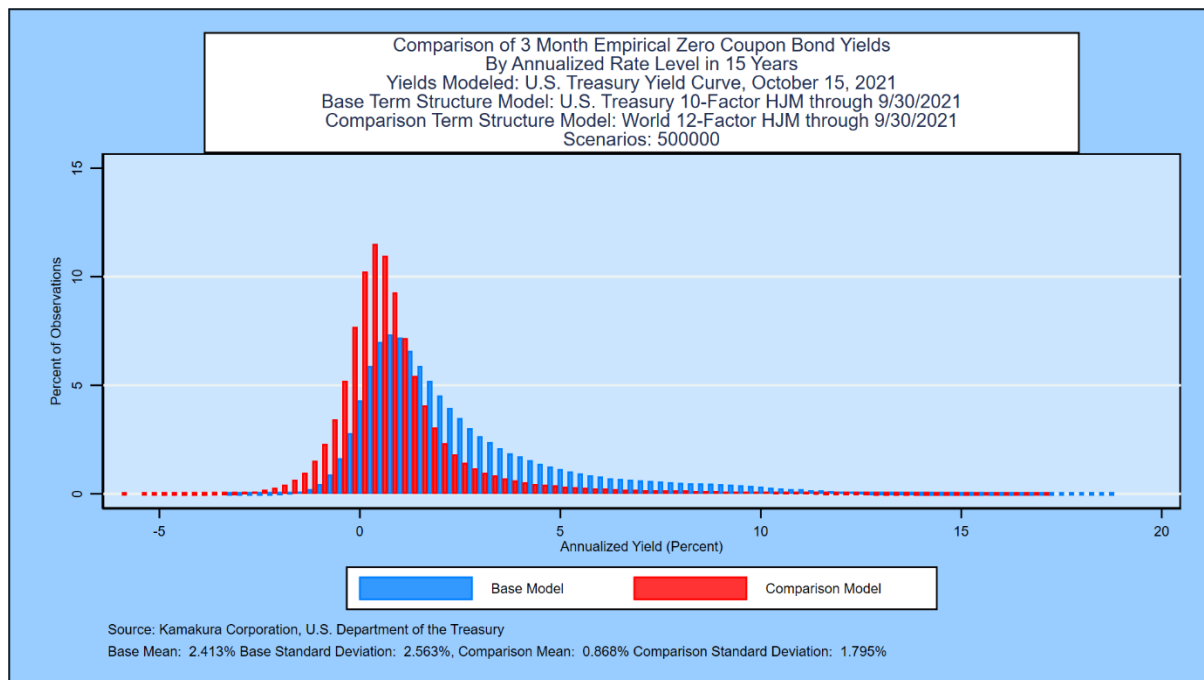
The pattern of previous Exhibits continues, as the World model at 15 years has significantly more negative rate observations and a tighter distribution than that generated from the U.S.-only model.

The simulated distribution of empirical 3-month rates shows the richness of the empirical drift function employed in both the World and U.S.-only HJM model. We stress test the derived coefficients in the Appendices.
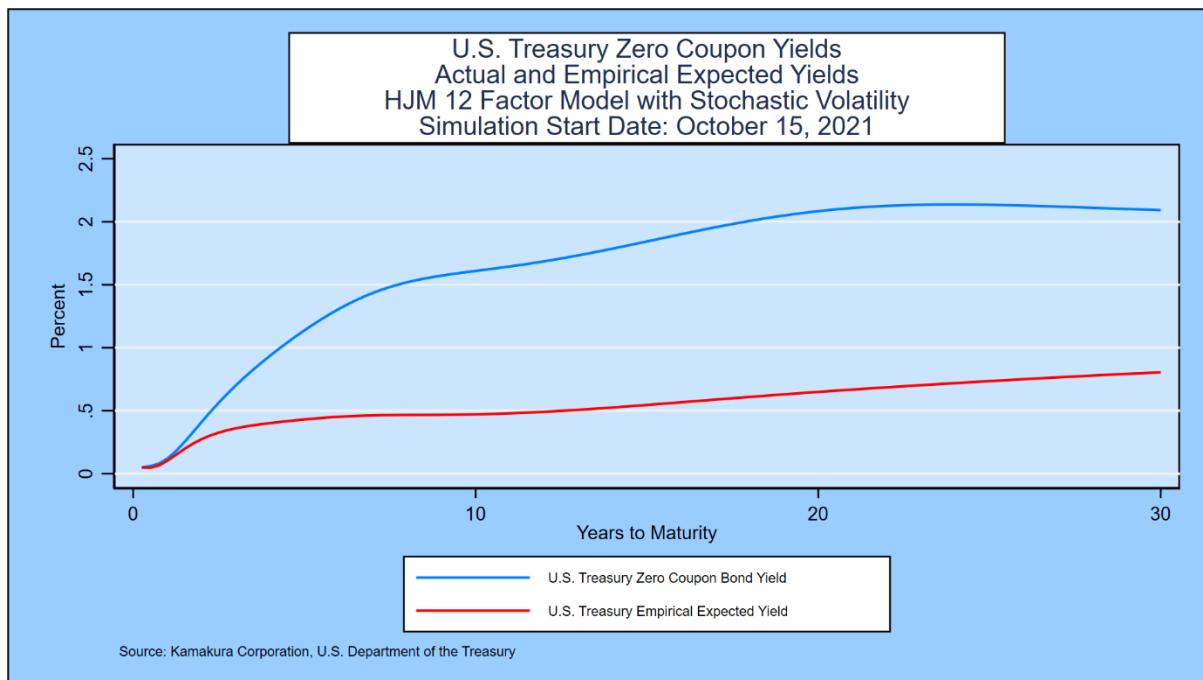
Time Series Distribution of Simulated Yields

We also plot the time series graphs of the mean, median, high, low, and various percentiles for empirical rates. We again seek to determine that there are no unexpected variations in the distribution of empirical yields over time. We can confirm that there are no unexpected variations in the distribution of empirical yields over time.

Simulation of the Term Premium

The size of the "term premium" of actual zero-coupon yields over the expected level of the short rate is a topic of great interest to both academics and policy makers. In a stochastic volatility model, the term premium must be determined by simulation because in general there is no closed-form solution for expected future rates. Exhibit X shows a much wider term premium for the U.S. using the World coefficients. The term premium widens significantly initially, and then it widens at a slower pace over longer maturities.

Exhibit X: Simulation of the Term Premium

U.S. Treasury Zero Coupon Yields
Actual and Empirical Expected Yields
HJM 12 Factor Model with Stochastic Volatility
Simulation Start Date: October 15, 2021

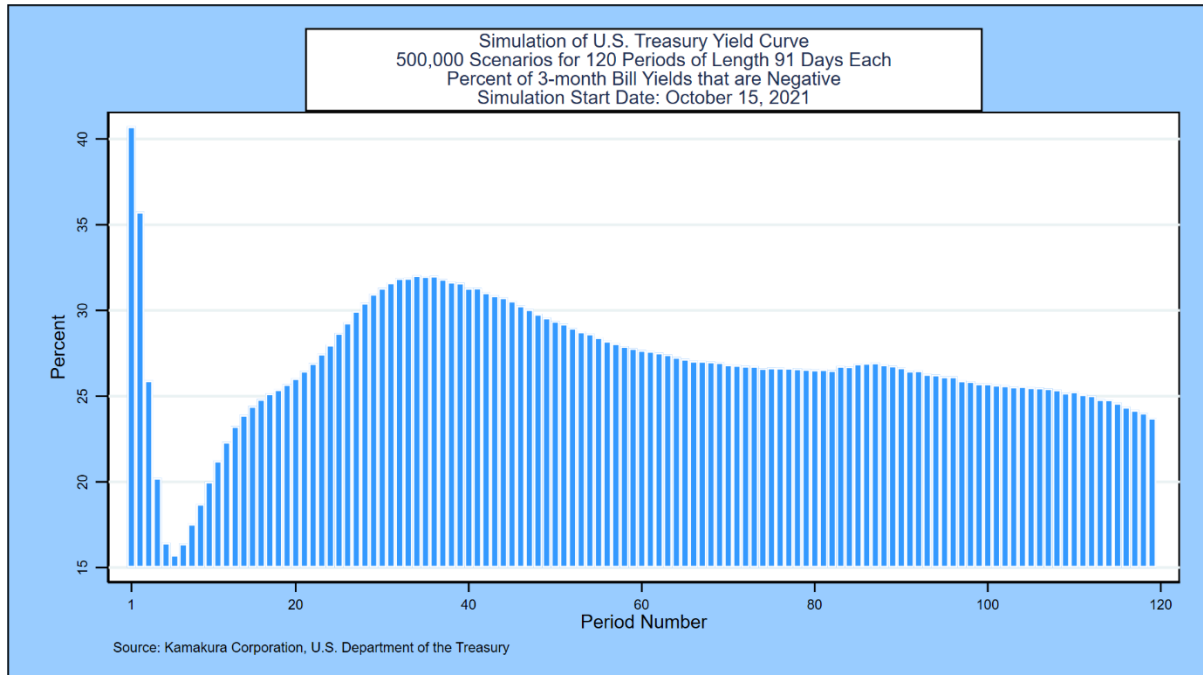Source: Kamakura Corporation, U.S. Department of the Treasury

We again conclude that the simulation produces results that are consistent with the "scientific knowledge" about the variation in interest rates around the world.

Probability of Negative Short-term Interest Rates

Exhibit XI shows the simulated probability of negative empirical 3-month Treasury bill rates at each 91-day time step using the World coefficients. Because of the sharp increase in forward rates embedded in the October 15, 2021 yield curve, the probability of negative rates drops initially and then rises to a level significantly above the probabilities generated using U.S.-only term structure model parameters.

Exhibit XI

Simulation of U.S. Treasury Yield Curve
500,000 Scenarios for 120 Periods of Length 91 Days Each
Percent of 3-month Bill Yields that are Negative
Simulation Start Date: October 15, 2021

Source: Kamakura Corporation, U.S. Department of the Treasury

IV. Conclusion

The 13-country government yield curves are driven by 12 factors, a number of factors very similar to the average for the single-country government yield studies.

The stochastic volatility assumption provides more accurate and more reasonable parameters than a constant volatility model, particularly in the context of Bayesian simulations as part of the model validation process. Exhibit XII summarizes the reasons for those conclusions:

Exhibit XII: Statistical Significance Summary and Volatility Classification
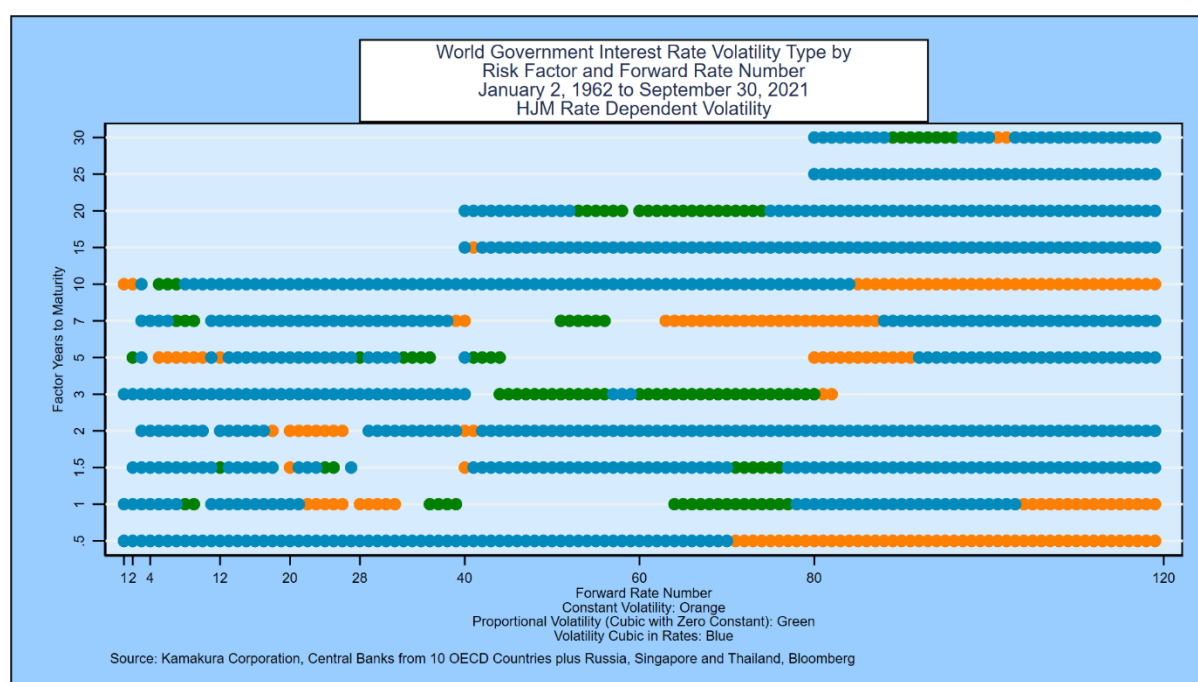
The vertical axis lists the maturities used as risk factors by years to maturity of the underlying quarterly forward rate. The risk factors are the idiosyncratic movement of each of these forward rates. If the risk factor is statistically significant in explaining the movement of forward rates with the quarterly maturities listed on the horizontal axis, a dot is placed in the grid. Note that the quarterly forward rates maturing in 15 and 20 years are only used as explanatory variables for maturities from 10 to 30 years. Similarly, the 25 and 30-year factors are used only for maturities over 20 years.
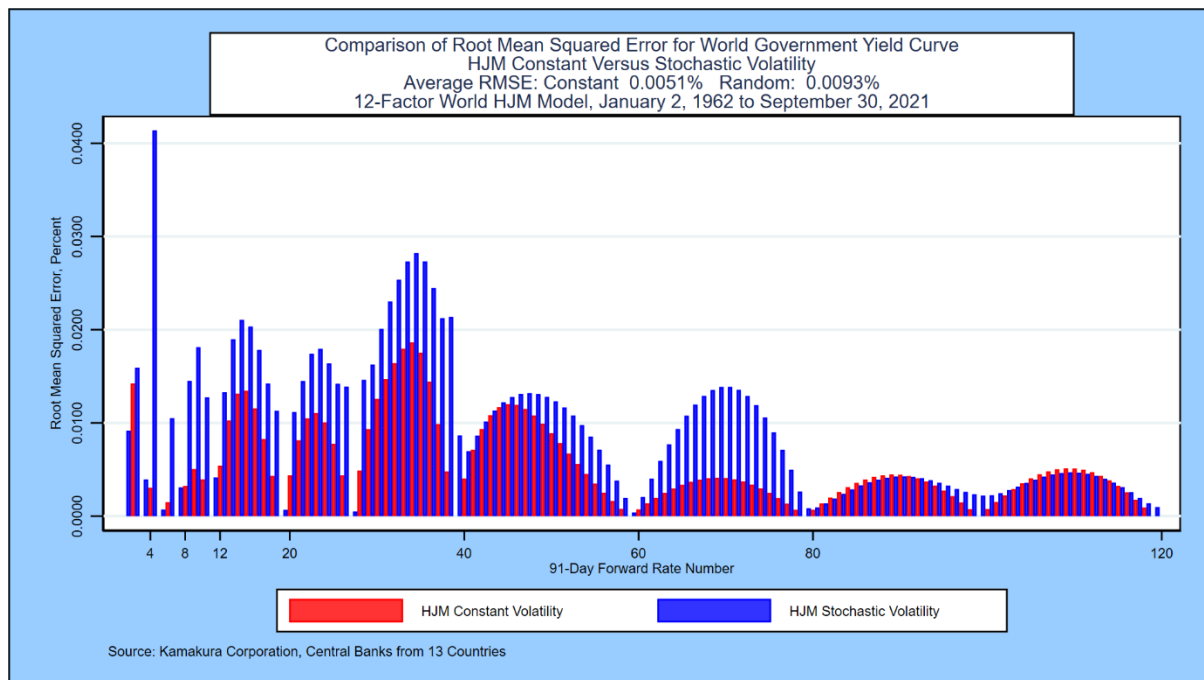
The nature of interest rate volatility for each combination of risk factor maturity and forward rate maturity is color coded. If the derived volatility is constant, the color code is orange. This is the affine specification. The graph shows immediately that a small minority of the risk factor maturity/forward rate maturity volatilities is consistent with the affine structure. The green and blue codes address the issue of whether interest rate volatility for that combination of risk factor maturity and forward rate maturity is zero or not when the forward rate is zero. If the measured volatility at a zero forward rate level is zero, the color code is green. Otherwise, the color code is blue. In both cases, the volatility is a stochastic function of the forward rates at the start of the simulation period.

The chart summarizes the fact that all 12 factors are statistically significant across the yield curve for the 13-country data set. The dominant derived interest rate volatility is the cubic stochastic volatility specification with a non-zero constant. An affine assumption for interest rate volatility is best fitting for a small minority of the combinations of risk factor maturity and forward rate maturity.

That being said, Exhibit XIII shows that the average root mean squared error for a 12-factor constant coefficient model is slightly below the average root mean squared error for the stochastic volatility model. Additional research on this point is forthcoming in subsequent editions of this paper.

Exhibit XIII: Root Mean Squared Error Comparison

Comparison of Root Mean Squared Error for World Government Yield Curve
HJM Constant Versus Stochastic Volatility
Average RMSE: Constant 0.0051%   Random: 0.0093%
12-Factor World HJM Model, January 2, 1962 to September 30, 2021

Root Mean Squared Error, Percent

91-Day Forward Rate Number

HJM Constant Volatility

HJM Stochastic Volatility

Source: Kamakura Corporation, Central Banks from 13 Countries

Appendix A: Single-Factor versus Multiple-Factor Models

In spite of the overwhelming evidence across countries that government bond yields are driven by multiple factors, the use of single-factor term structure models in interest rate risk management systems remains common even in some of the world's largest banks. This appendix asks and answers a number of important questions on the use of one-factor models that any sophisticated model audit would pose. Given the answers below, most analysts would conclude that one-factor term structure models are less accurate than a long list of multi-factor term structure models and that the one-factor models would therefore fail a model audit.
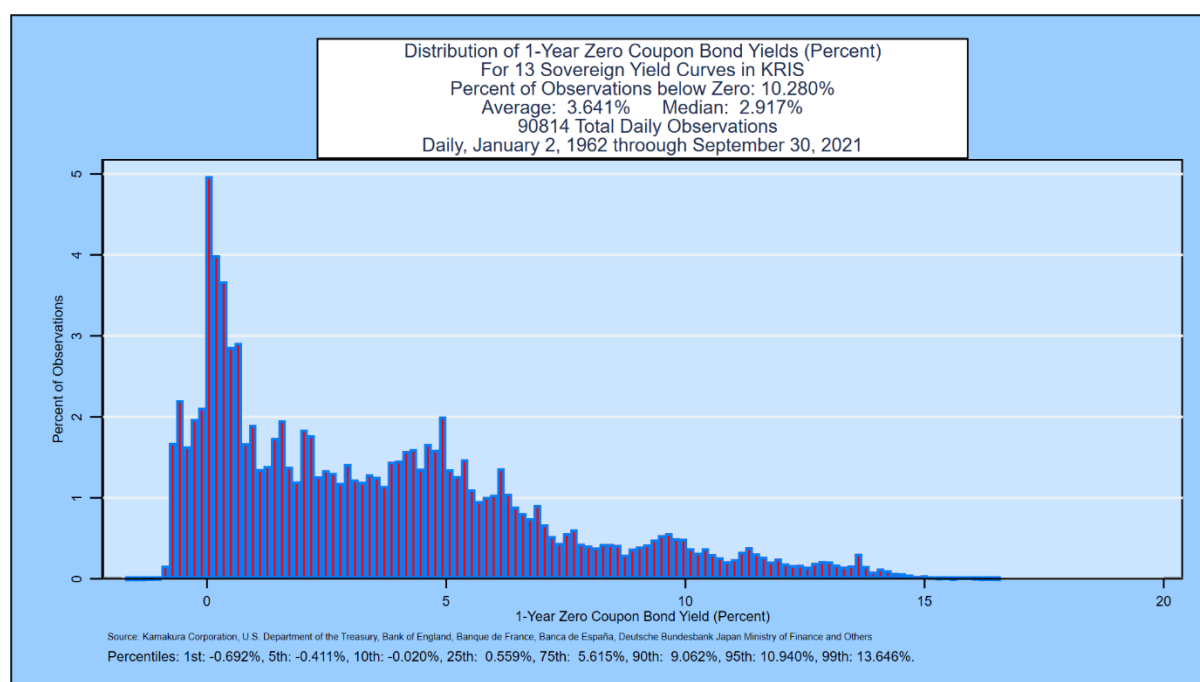
We address two classes of one-factor term structure models, all of which are special cases of the Heath, Jarrow, and Morton framework, in this appendix using data from the 13-country data set.

- One-factor models with rate-dependent interest rate volatility:
    - Cox, Ingersoll, and Ross (1985)
    - Black, Derman and Toy (1990)
    - Black and Karasinski (1991)

- One-factor models with constant interest rate volatility (affine models):
    - Vasicek (1977)
    - Ho and Lee (1986)
    - Extended Vasicek or Hull and White Model (1990, 1993)

Non-parametric test 1: Can interest rates be negative in the model?

The one-factor models with rate-dependent interest rate volatility make it impossible for interest rates to be negative without *ad hoc* shifts in the simulated yield curves. Is this implication true or false? It is false, as this histogram of 1-year zero-coupon bond yields from government securities in 13 countries confirms. 10.280% of observations were negative:
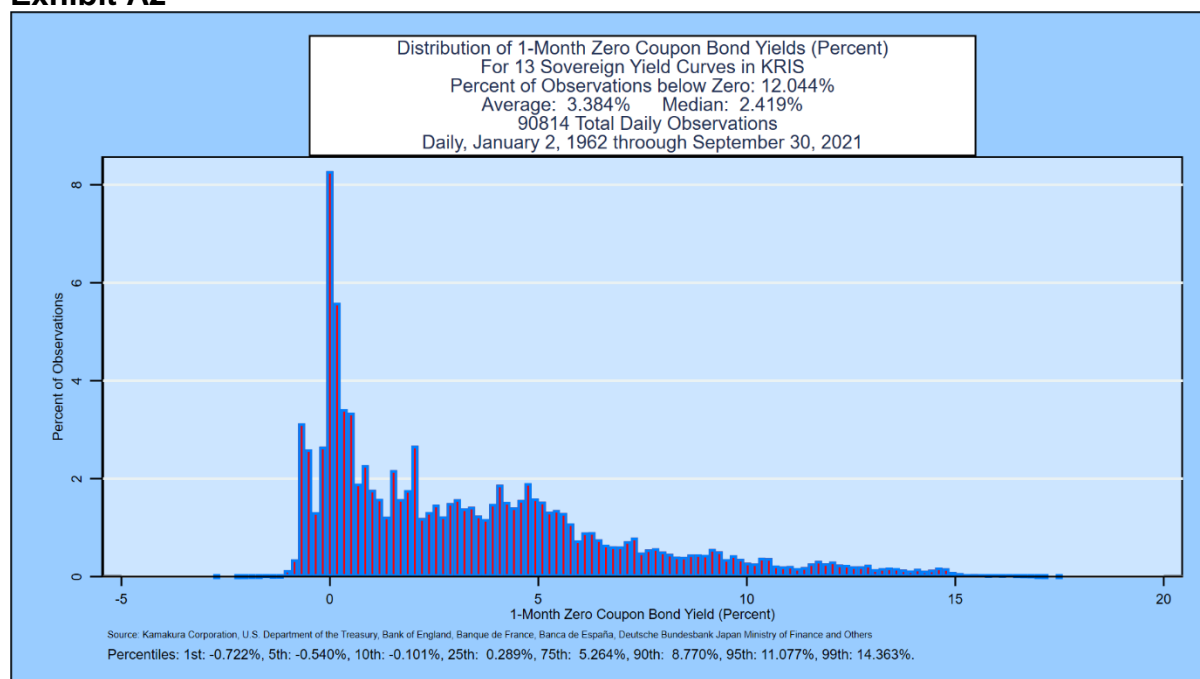
Exhibit A1

Distribution of 1-Year Zero Coupon Bond Yields (Percent)
For 13 Sovereign Yield Curves in KRIS
Percent of Observations below Zero: 10.280%
Average: 3.641%   Median: 2.917%
90814 Total Daily Observations
Daily, January 2, 1962 throough September 30, 2021

Source: Kamakura Corporation, U.S. Department of the Treasury, Bank of England, Banque de France, Banca de España, Deutsche Bundesbank Japan Ministry of Finance and Others
Percentiles: 1st: -0.692%, 5th: -0.411%, 10th: -0.020%, 25th: 0.559%, 75th: 5.615%, 90th: 9.062%, 95th: 10.940%, 99th: 13.646%.

Non-parametric test 2: As commonly implemented, one-factor term structure models imply that all yields will either (a) rise, (b) fall, or (c) remain unchanged.   This implication is false, as documented for the 13-country data set in Table II. In fact, yield curves have twisted on 85.21% of the observations for the World government markets studied.

Non-parametric test 3: The constant coefficient one-factor models imply that zero-coupon yields are normally distributed and so are the changes in zero-coupon yields. In each of the 13 individual country studies, this implication is rejected by three common statistical tests for 120 of 120 quarterly maturities for zero yields and for all 120 of the quarterly changes. This histogram of 1-month zero-coupon bond yields from consolidated zero-coupon bond yields in 13 countries provides simple visual confirmation that normality is a poor approximation to the probability distribution of interest rates.

Exhibit A2

Distribution of 1-Month Zero Coupon Bond Yields (Percent)
For 13 Sovereign Yield Curves in KRIS
Percent of Observations below Zero: 12.044%
Average: 3.384%   Median: 2.419%
90814 Total Daily Observations
Daily, January 2, 1962 through September 30, 2021

Source: Kamakura Corporation, U.S. Department of the Treasury, Bank of England, Banque de France, Banca de España, Deutsche Bundesbank Japan Ministry of Finance and Others

Percentiles: 1st: -0.722%, 5th: -0.540%, 10th: -0.101%, 25th: 0.289%, 75th: 5.264%, 90th: 8.770%, 95th: 11.077%, 99th: 14.363%.

Assertion A: There are no factors other than the short-term rate of interest that are statistically significant in explaining yield curve movements.

This assertion is false. Table V shows, using principal components analysis, that 12-17 factors are needed to explain the movements of the 13-country yield dataset. Exhibit XII makes the same point in more detail.

Assertion B: There may be more than one factor, but the incremental explanatory power of the 2nd and other factors is so miniscule as to be useless.

This assertion is false, as the 2nd through 12th factors in the 13-country data set explain 38.03% of forward rate movements, compared to 61.97% for the first factor alone. In most countries, the best "first factor" is not the short rate of interest used by many large banks; it is the parallel shift factor of the Ho and Lee model.

Assertion C: A one-factor "regime shift" model is all that is necessary to match the explanatory power of the 2nd and other factors.

This assertion is also false. A recent study prepared for a major U.S. bank regulator also confirmed that a one-factor "regime shift" term structure model made essentially no incremental contribution toward resolving the persistent lack of accuracy in one-factor term structure models.

Finally, a Kamakura Corporation study comparing a 1-factor and 10-factor Heath Jarrow and Morton simulation results shows that the simulated volatility of interest rates is underestimated by 61% to 83% in the one-factor model. Moreover, the average level of simulated U.S. Treasury yields is biased lower. Finally, the probability of negative rates implied by the one-factor model is also biased low as shown here:

Table A1

| Zero Coupon Bond Yields							
Yield Maturity	Simulated Years Forward	Base Mean Yield	Comparison Mean Yield	Absolute Difference	Base Standard Deviation	Compare Standard Deviation	Percent Difference
3 months	1	0.076	0.083	0.007	0.232	0.090	-61.2%
3 months	5	1.041	0.887	-0.154	1.003	0.247	-75.4%
3 months	10	1.588	1.114	-0.474	1.866	0.426	-77.2%
3 months	15	2.151	1.283	-0.868	2.580	0.580	-77.5%
5 years	1	1.244	1.238	-0.006	0.494	0.082	-83.4%
5 years	5	2.193	1.990	-0.203	1.190	0.254	-78.7%
5 years	10	2.815	2.299	-0.516	1.790	0.378	-78.9%
5 years	15	3.446	2.480	-0.966	2.349	0.452	-80.8%
10 years	1	1.922	1.913	-0.009	0.570	0.131	-77.0%
10 years	5	2.760	2.561	-0.199	1.160	0.271	-76.6%
10 years	10	3.468	2.897	-0.571	1.691	0.309	-81.7%
10 years	15	4.100	3.080	-1.020	2.136	0.343	-83.9%
Par Coupon Bond Yields							
3 months	1	0.076	0.083	0.007	0.232	0.090	-61.2%
3 months	5	1.044	0.888	-0.156	1.009	0.247	-75.5%
3 months	10	1.596	1.116	-0.480	1.887	0.427	-77.4%
3 months	15	2.165	1.286	-0.879	2.615	0.584	-77.7%
5 years	1	1.235	1.230	-0.005	0.488	0.082	-83.2%
5 years	5	2.189	1.984	-0.205	1.196	0.255	-78.7%
5 years	10	2.816	2.292	-0.524	1.821	0.382	-79.0%
5 years	15	3.459	2.475	-0.984	2.408	0.461	-80.9%
10 years	1	1.874	1.867	-0.007	0.551	0.124	-77.5%
10 years	5	2.711	2.513	-0.198	1.147	0.266	-76.8%
10 years	10	3.405	2.844	-0.561	1.695	0.316	-81.4%
10 years	15	4.401	3.024	-1.377	2.181	0.359	-83.5%

Appendix B: Tests of Empirical Drift and Stress Tests of Rate Volatility

A determination that out-of-sample simulations will be "consistent with scientific knowledge of the problem" can in fact be gleaned without a full simulation. One practical modeling objective is to obtain out-of-sample rate scenarios that "do not blow up," rising or falling to extreme values that are highly unlikely to occur in reality. Heath, Jarrow, and Morton make it clear how this objective is achieved in an HJM model:

1. First of all, rate shocks and rate volatilities are the same for the simulation of both risk neutral and empirical rates.
2. This requires that the interest rate volatility functions be capped to prevent "explosions" in rate simulations, as we have done above.
3. HJM also make it clear that the risk neutral drift in each segment of the yield curve is *completely determined* by the choice of volatility functions and the slope of the initial yield curve. This insight is not apparent to many practitioners because the early work of Vasicek [1977] and Hull and White 3 decades ago seems to embed mean reversion in risk neutral interest rates.
4. The drift in empirical rates, however, is determined econometrically. In the model above the drift function employed adds a cubic function of the corresponding forward rate to the base drift that applies when rates are zero or below.

The Bayesian "scientific knowledge" of empirical rate movements leads us to expect some mean reversion in rates. When rates are high, we expect the empirical drift will be negative in each section of the empirical yield curve. When rates are low, we expect that the empirical drift to be positive. By applying hypothetical flat yield curves from zero (or less) to 15%, we can confirm that empirical drift in the U.S. Treasury stand-alone HJM model is consistent with these expectations.
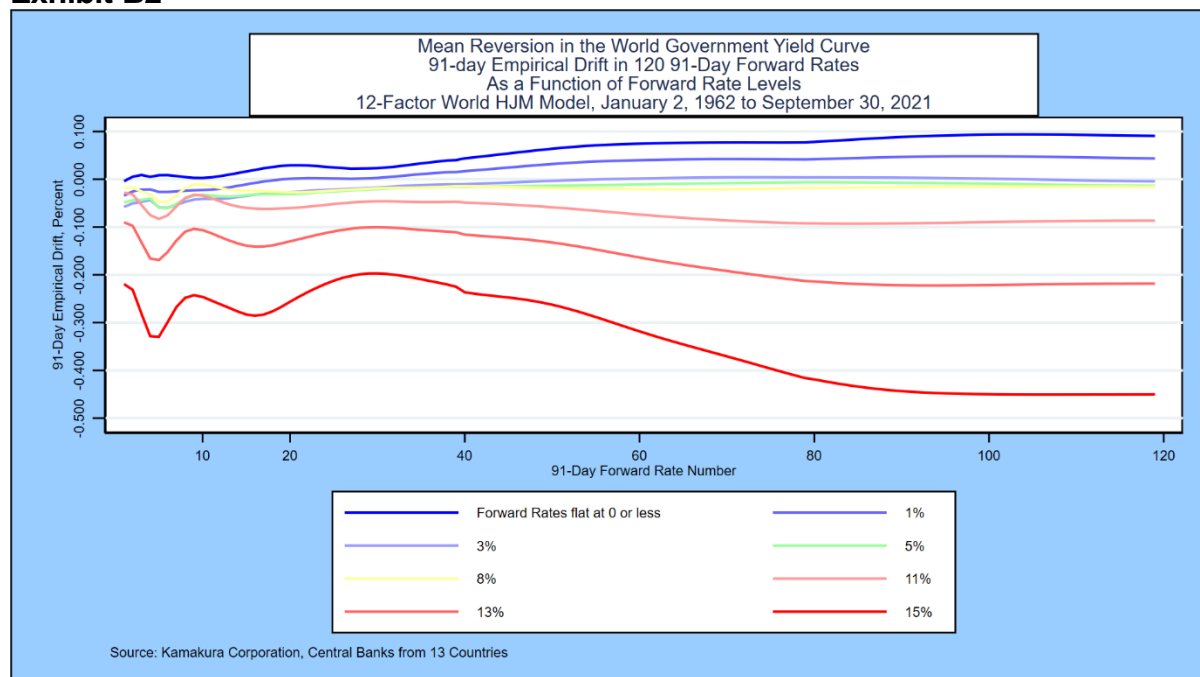
Exhibit B1
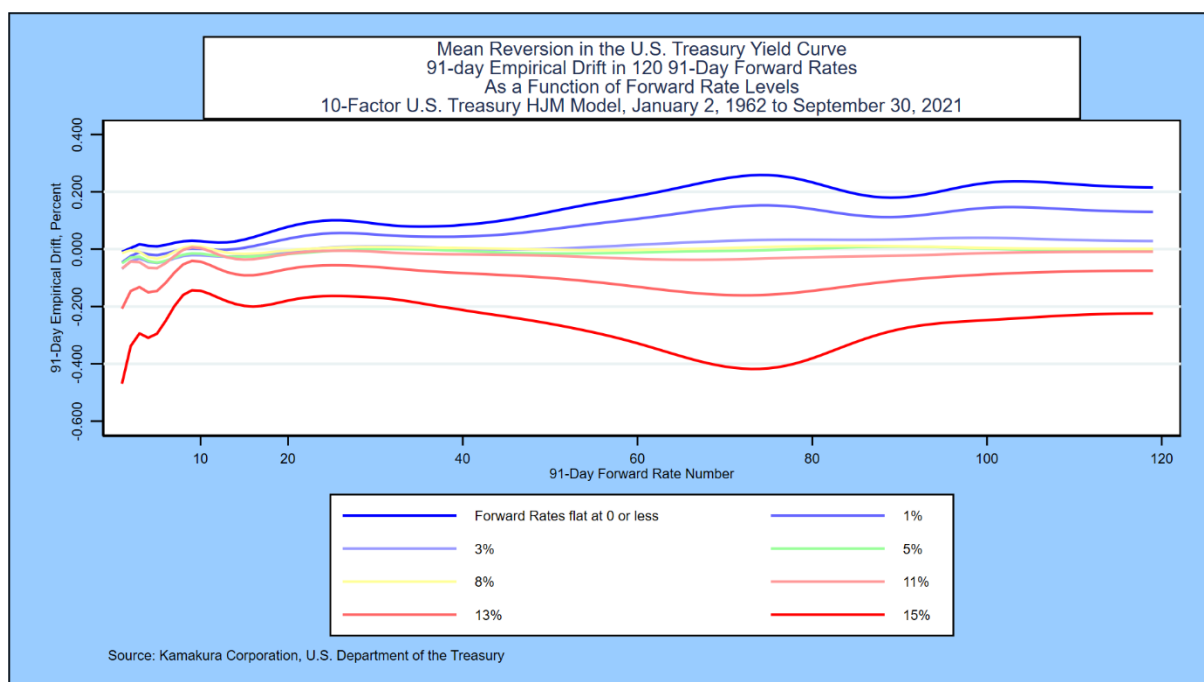
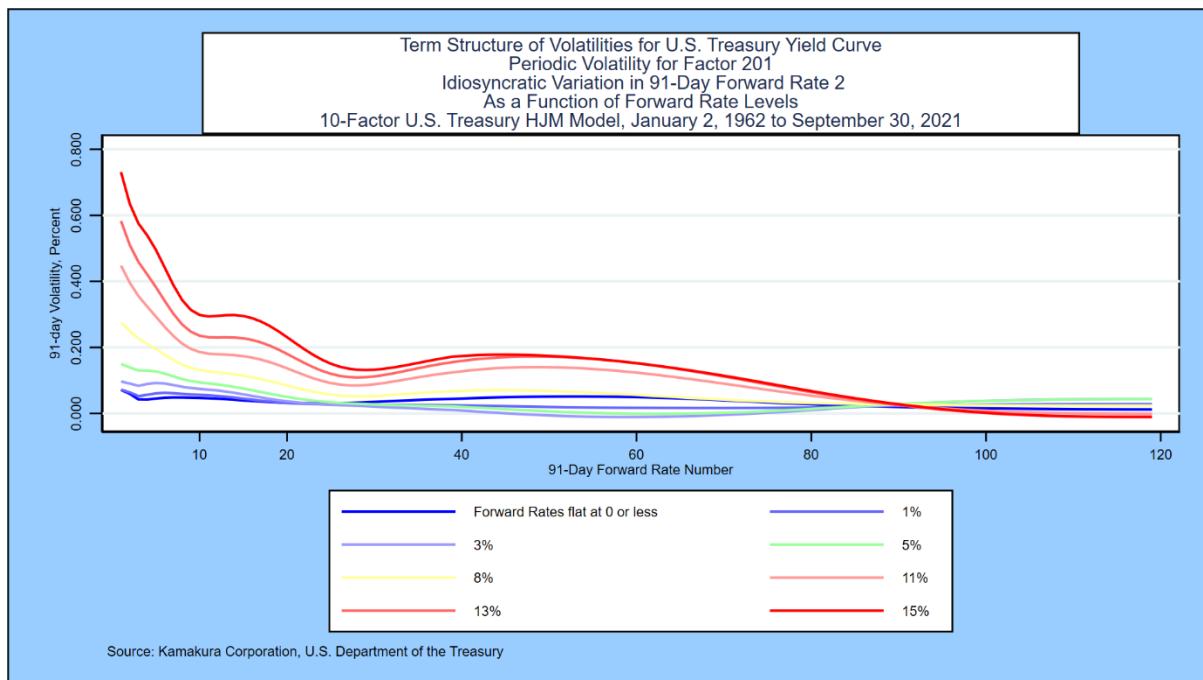The empirical drift for the World model through September 30, 2021 is shown in Exhibit B2.

Exhibit B2

Mean Reversion in the World Government Yield Curve
91-day Empirical Drift in 120 91-Day Forward Rates
As a Function of Forward Rate Levels
12-Factor World HJM Model, January 2, 1962 to September 30, 2021

Legend:
- Forward Rates flat at 0 or less
- 1%
- 3%
- 5%
- 8%
- 11%
- 13%
- 15%

Source: Kamakura Corporation, Central Banks from 13 Countries

Simply by visual inspection, it is clear that the fitted drift in empirical rates for the World model shows mean reversion that is reasonably similar to that fitted to U.S. data alone.
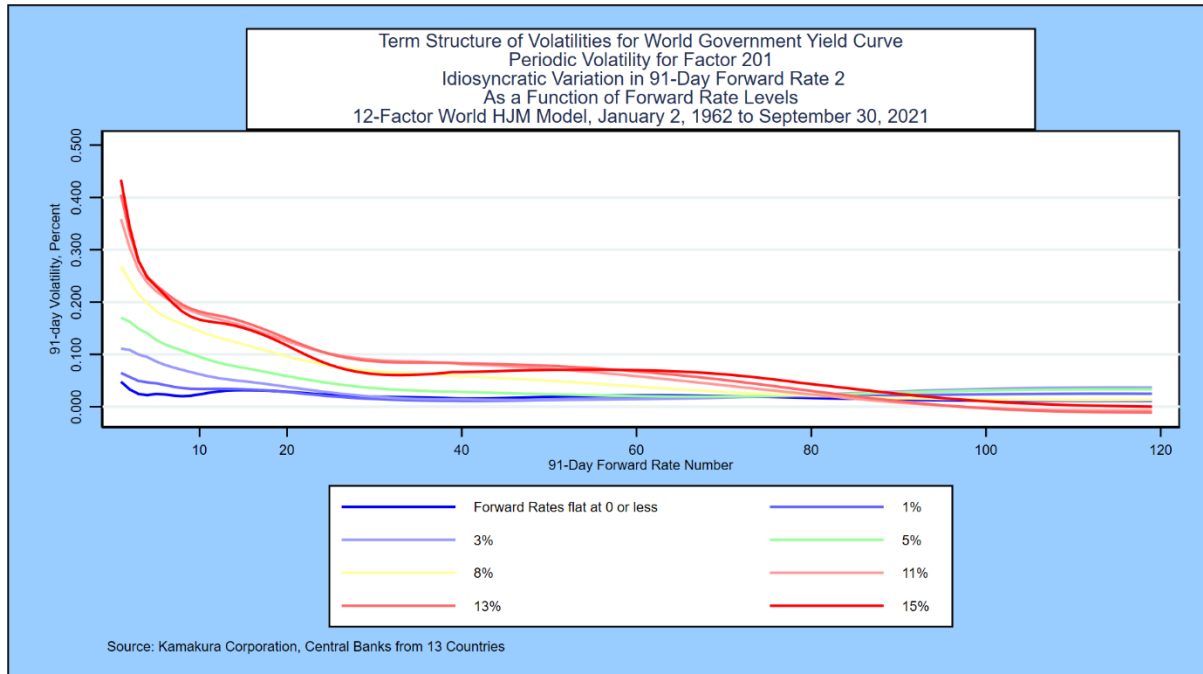
We can apply the same tests to the interest rate volatility functions to confirm their robustness and that they do not "blow up" at either very high or very low-rate levels. The first factor in the model above is the idiosyncratic movement in the first random forward rate segment. We expect this first factor to have volatility that declines with maturity, and which is higher when rates are high. The graph below shows that those expectations are met by U.S. Treasury single-country data:

Exhibit B3

Term Structure of Volatilities for U.S. Treasury Yield Curve
Periodic Volatility for Factor 201
Idiosyncratic Variation in 91-Day Forward Rate 2
As a Function of Forward Rate Levels
10-Factor U.S. Treasury HJM Model, January 2, 1962 to September 30, 2021

Source: Kamakura Corporation, U.S. Department of the Treasury

We again compare the U.S. stand-alone model movements in interest rate volatility for the first factor with World experience, shown in Exhibit B4:
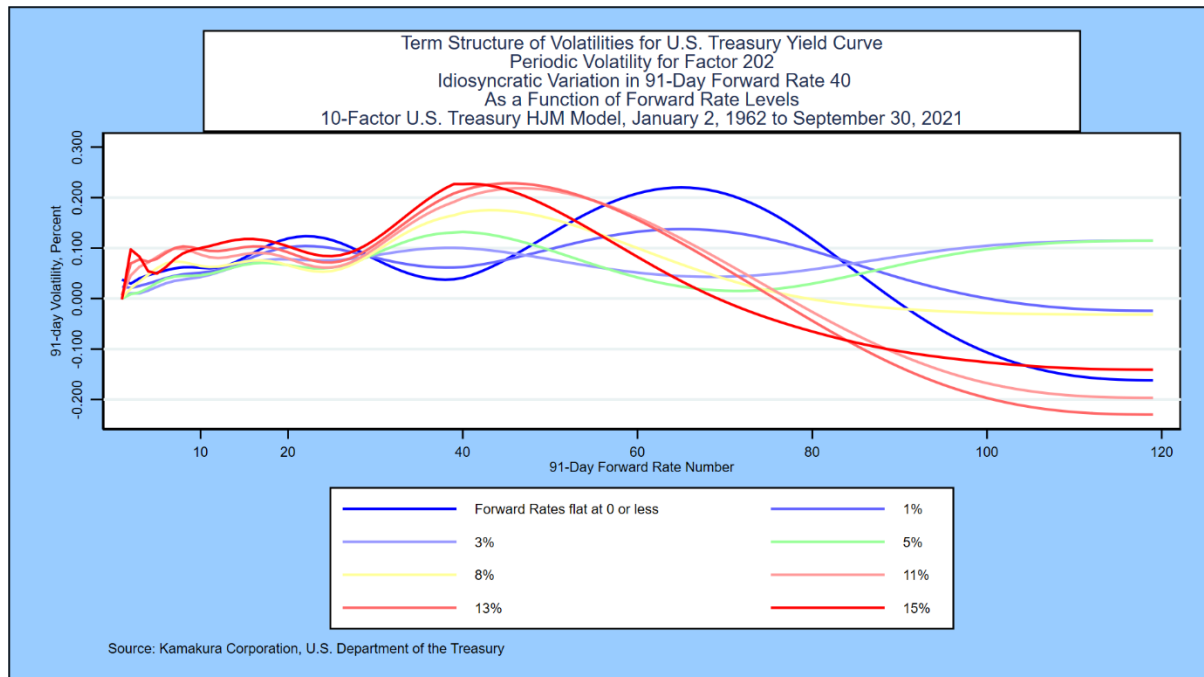
Exhibit B4

Term Structure of Volatilities for World Government Yield Curve
Periodic Volatility for Factor 201
Idiosyncratic Variation in 91-Day Forward Rate 2
As a Function of Forward Rate Levels
12-Factor World HJM Model, January 2, 1962 to September 30, 2021

Source: Kamakura Corporation, Central Banks from 13 Countries

The World interest rate volatility again shows a stress-tested pattern that is both similar to U.S. data alone and consistent with expectations.
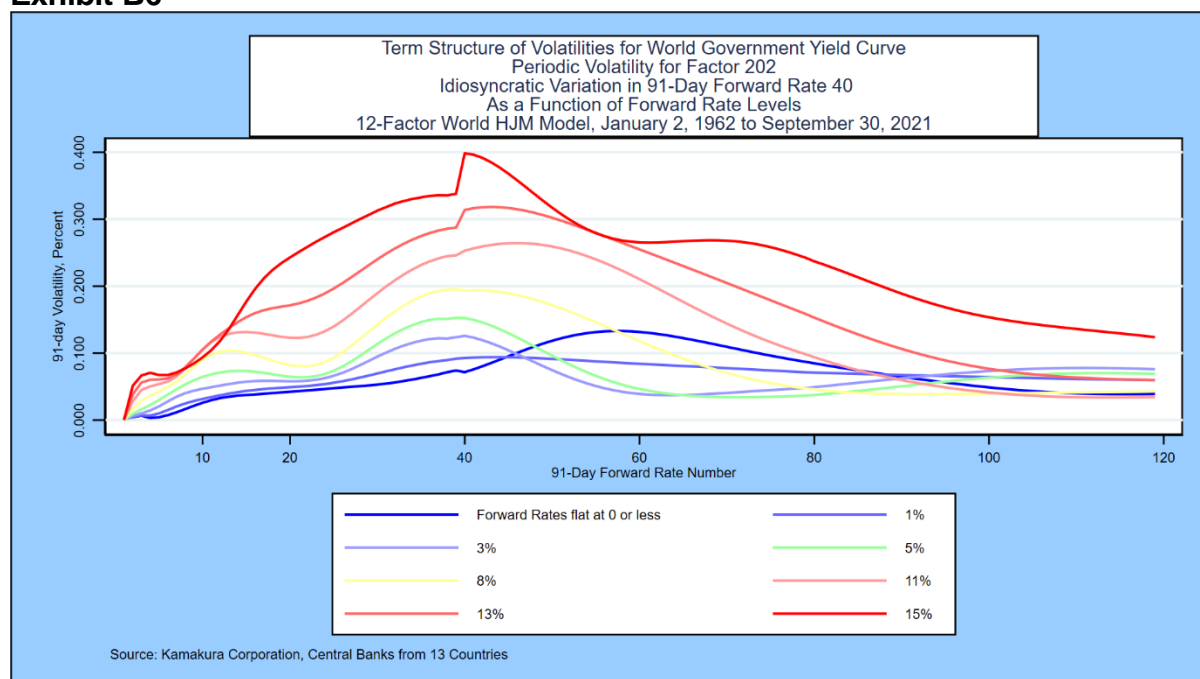
The second factor in the model is the idiosyncratic variation in the 91-day forward rate segment maturing in 10 years. Because this point is near the middle of the yield curve, we expect volatility functions can be both positive and negative. Our main concern is that volatility does not blow up as we stress test the level of the forward rate curve. Again, our expectations are met, as shown below for the U.S. stand-alone model.

Exhibit B5

Term Structure of Volatilities for U.S. Treasury Yield Curve
Periodic Volatility for Factor 202
Idiosyncratic Variation in 91-Day Forward Rate 40
As a Function of Forward Rate Levels
10-Factor U.S. Treasury HJM Model, January 2, 1962 to September 30, 2021

Source: Kamakura Corporation, U.S. Department of the Treasury

How does this compare with volatility for the second factor (which is also the idiosyncratic movements in the 91-day forward rate maturing in 10 years) for the World model? The answer is given in Exhibit B6:

Exhibit B6

Term Structure of Volatilities for World Government Yield Curve
Periodic Volatility for Factor 202
Idiosyncratic Variation in 91-Day Forward Rate 40
As a Function of Forward Rate Levels
12-Factor World HJM Model, January 2, 1962 to September 30, 2021

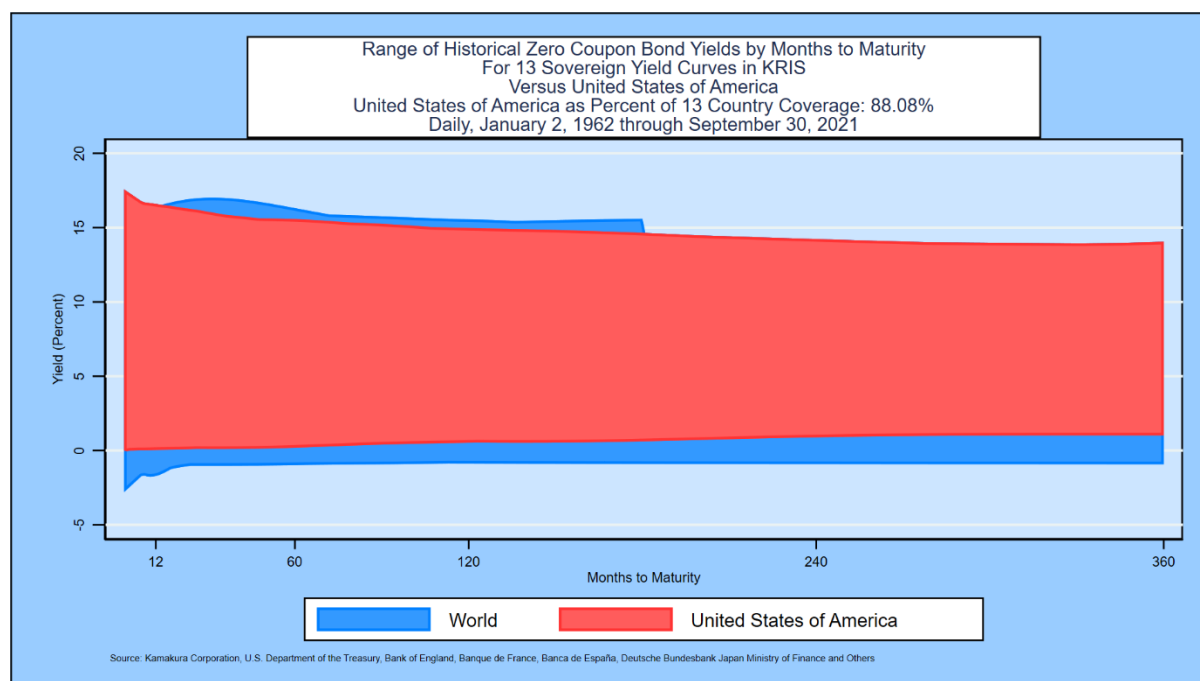Source: Kamakura Corporation, Central Banks from 13 Countries

Here, the results of the World model differ from the U.S.-only model. An upward shock in the 10-year forward rate drives up yields in the middle of the World yield curve, with smaller impacts on the short end and long end of the curve. Generally speaking, volatility increases as the level of the 10-year forward rate rises. This pattern is more consistent with the author's expectations than the U.S.-only results.

Appendix C: Comparing the Span of Interest Rates with the World Data Set

A common econometric challenge is fitting a model to the data at hand and measuring the risk that the model behaves badly out of sample. One measure of this risk for each single-country data set is to compare the range of interest rates experienced with the consolidated13-country World data set over maturities to 30 years. We use monthly zero-coupon bond yields for these calculations. The results are shown in Exhibit C1 for the U.S. stand-alone data set and the consolidated World data set:

Exhibit C1

Range of Historical Zero Coupon Bond Yields by Months to Maturity
For 13 Sovereign Yield Curves in KRIS
Versus United States of America
United States of America as Percent of 13 Country Coverage: 88.08%
Daily, January 2, 1962 through September 30, 2021

Source: Kamakura Corporation, U.S. Department of the Treasury, Bank of England, Banque de France, Banca de España, Deutsche Bundesbank Japan Ministry of Finance and Others

The U.S. data set spans 88.08% of World interest rate experience. This is the highest proportion of any of the 13 countries included in the World data set. Even in the U.S. case, however, using the coefficients of the World model would provide greater realism in simulations when rates turn negative than a stand-alone U.S. model and perhaps more realistic responses of volatility to forward rates levels as we saw in Exhibits VI to IX.

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