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A 10 Factor Heath, Jarrow and Morton Stochastic Volatility Model for the U.S. Treasury Yield Curve, Using Daily Data from January 1, 1962 through December 31, 2020

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First Version: March 9, 2021
This Version: March 9, 2021

ABSTRACT

This paper analyzes the number and the nature of factors driving the movements in the U.S. Treasury yield curve from January 2, 1962 through December 31, 2020. The process of model implementation reveals a number of important insights for interest rate modeling generally. First, model validation of historical yields is important because those yields are the product of a third-party curve fitting process that may produce spurious indications of interest rate volatility. Second, quantitative measures of smoothness and international comparisons of smoothness provide a basis for measuring the quality of simulated yield curves. Third, we outline a process for incorporating insights from the Japanese and European experience with negative interest rates into term structure models with stochastic volatility in the United States and other countries. Finally, we illustrate the process for comparing stochastic volatility and affine models of the term structure. We conclude that stochastic volatility models have a superior fit to the history of yield movements in the U.S. Treasury market.

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A 10 Factor Heath, Jarrow and Morton Model for the U.S. Treasury Yield Curve, Using Daily Data from January 1, 1962 through December 31, 2020

Government yield curves are a critical input to the risk management calculations of central banks, bank regulators, major banks, insurance firms, fund managers, pension funds, and endowments around the world. With the internationalization of fixed income investing, it is important to understand the dynamics of movements in yield curves world-wide, in addition to the major bond markets like those in Frankfurt, London, New York and Tokyo. In this paper, we fit a multi-factor Heath, Jarrow and Morton model to daily data from the U.S. Treasury market over the period from January 2, 1962 to December 31, 2020. The modeling process reveals a number of important implications for term structure modeling in other government bond markets.

Section I discusses the origin and characteristics of the daily data base of U.S. Treasury yields provided by the U.S. Department of the Treasury. We discuss yield curve smoothness and volatility fitting as measures for judging the quality of government-generated yield curve time series, including U.S. Treasury yield curves. We conclude that the U.S. Department of Treasury time series is realistically smooth and a reliable foundation for term structure modeling. This compares with recent findings from Australia, Canada, Japan, and Thailand where we found that yield curves were unrealistically rough and that modification of the input data was necessary for a realistic model, a standard part of a Bayesian model validation process.

Section II outlines the process for determining whether the interest rate volatility for the factors driving the U.S. Treasury yield curve is constant (an "affine" model) or stochastic, typically expressed as a function of the level of interest rates. We note the extensive experience with negative interest rates in the European and Japanese government bond markets and use insights from that experience in fitting volatility in the U.S. Treasury market. Section III describes the process of fitting five different Heath, Jarrow and Morton models to U.S. Treasury yield data: models with 1, 2, 3, 6 and 10 factors. We conclude Section III with extensive Bayesian model validation procedures based on a 30-year forward-looking simulation of 500,000 scenarios. Section IV concludes the paper. The Appendix illustrates a sample model validation process for widely used one factor term structure models using U.S. data.

I. U.S. Treasury Data: Special Characteristics

A multi-factor term structure model is the foundation for best practice asset and liability management, market risk, economic capital, interest rate risk in the banking book, stress-testing and the internal capital adequacy assessment process. The objective in this paper is to illustrate the derivation of a multi-factor Heath Jarrow and Morton model of the U.S. Treasury yield curve. As a by-product, the analysis has the potential to detect common data problems associated with yield curve histories and employs a standard methodology for quantification and resolution of those problems. Previous implementations of multi-factor Heath, Jarrow and Morton models have covered the following bond market sectors:

Australia	Commonwealth Government Securities
Canada	Government of Canada Securities

Germany	German Bunds
Japan	Japanese Government Bonds
Singapore	Singapore Government Securities
Spain	Spanish Government Bonds
Sweden	Swedish Government Securities
Thailand	Thai Government Securities
United Kingdom	United Kingdom Government Bonds
United States	U.S. Treasury Securities

The first step in data model validation for the U.S. Treasury market is to examine the historical availability of bond yields over time. This availability is summarized in Table I.

Table I

Kamakura Corporation
HJM 10 Factor Model
U.S. Treasury
Using Daily Data from January 1, 1962 through December 31, 2020
Date of Analysis: December 31, 2020
Number of Observations by Data Regime

Data Regime	Start Date	End Date	Number of Observations
USA: 1, 3, 5 and 10 years	1/2/1962	6/30/1969	1,870
USA: 1, 3, 5, 7 and 10 years	7/1/1969	5/28/1976	1,722
USA: 1, 2, 3, 5, 7 and 10 years	6/1/1976	2/14/1977	178
USA: 1, 2, 3, 5, 7, 10 and 30 years	2/15/1977	12/31/1981	1,213
USA: 3 and 6 months with 1, 2, 3, 5, 7, 10 and 30 years	1/4/1982	9/30/1993	2,935
USA: 3 and 6 months with 1, 2, 3, 5, 7, 10, 20 and 30 years	10/1/1993	7/30/2001	1,960
USA: 1, 3 and 6 months with 1, 2, 3, 5, 7, 10, 20 and 30 years	7/31/2001	2/15/2002	135
USA: 1, 3 and 6 months with 1, 2, 3, 5, 7, 10, and 20 years	2/19/2002	2/8/2006	994
USA: Second era: 1, 3 and 6 months with 1, 2, 3, 5, 7, 10, 20 and 30 years	2/9/2006	3/29/2018	3,041
USA: 1 day, 1, 3 and 6 months with 1, 2, 3, 5, 7, 10, 20 and 30 years	4/2/2018	12/31/2020	689
Total Observations			14,737

The data shows that the U.S. Treasury's data history is typical in its occasional changes in "data regime," i.e., which of the maturities are available on a given date. On April 2, 2018, the Federal Reserve Bank of New York began reporting the overnight SOFR rate on a daily basis. Beginning on October 16, 2018, the U.S. Treasury added the 8-week Treasury bill yield to its daily reporting.

Because our Heath, Jarrow and Morton analysis makes use of a yield curve with 91-day (quarterly) forward rate segments, the next step in data model validation is to fit quarterly forward rates to the raw coupon-bearing bond yields. The smoothness of the resulting forward rates will be a function of both the quality of the raw data from a smoothness point of view and the smoothness implied by the secondary smoothing process. To ensure the maximum smoothness from the secondary smoothing process, we use the maximum smoothness forward rate methodology of Adams and van Deventer [1994], as corrected in van Deventer and Imai [1996]. Adams and van Deventer show that the maximum smoothness method overcomes the problems of the cubic spline approach of McCulloch, and, unlike the Svensson [1994] approach, allows for a perfect fit to the raw data provided by the U.S. Department of the Treasury. See Jarrow [2014] for information on the problems with Svensson yield curve fitting.

We then conduct a visual inspection of the resulting forward rates implied by the raw data. A video of the daily quarterly forward rates (in red) versus the zero-coupon bond yields (blue) implied by the U.S. Treasury data on every business day from 1962 through December 31, 2018 is given here:

https://www.youtube.com/watch?v=aHxqTCWwPNg&t=23s

The video will be updated through 2020 in coming weeks. The smoothness of the quarterly forward rate curve can be measured quantitatively using the quarterly forward rates implied by the U.S. Treasury yield curves. For a yield curve that consists of N quarterly forward rates, the discrete smoothness statistic at time t $Z_N(t)$ is the sum of the squared second differences in the forward rates, as explained by Adams and van Deventer [1994]. A closed form continuous smoothing statistic can also be calculated when the functional form of the continuous forward rate is known. The discrete smoothness statistic is given here:

$$Z_N(t) = \sum_{i=3}^{N} \left[\left(f_i(t) - f_{i-1}(t) \right) - \left(f_{i-1}(t) - f_{i-2}(t) \right) \right]^2$$

A statistical comparison of smoothness for unmodified Japan Ministry of Finance data with data from the U.S. Department of the Treasury, both smoothed using the maximum smoothness forward rate approach, confirms that the first half of the Japanese Government Bond forward rate data set is much more volatile than the U.S. data, as the video shows. This video makes the yield curve comparison from 1974 to 2016 on a daily basis:

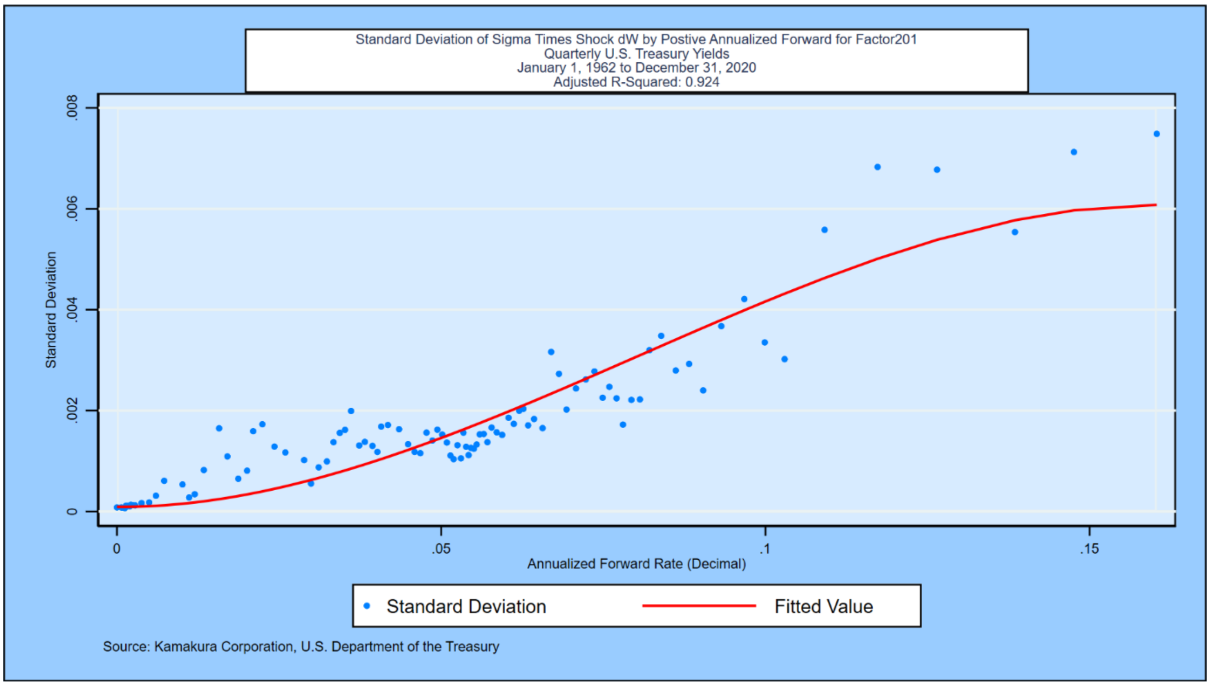
https://www.youtube.com/watch?v=h_A1yYBQZ2c&list=PLFtDZOVCnk_oKk8bC4OnwQ_U94MCu9_EL&index=1

We conclude that the raw data provided by the Japan Ministry of Finance implies unrealistic movements in forward rates. The U.S. Treasury series, however, is realistically smooth and we use that data with confidence in what follows. We apply a final screen for outliers in the process of deriving the stochastic volatility functions.

II. Constant versus Stochastic Volatility

Constant volatility ("affine") term structure models are commonly used for their ease of simulation and estimation of "future expected rates" in order to determine the "term premium" in current yields. Prominent examples are Adrian, Crump and Moench [2013], Kim and Wright [2005], and Duffie and Kan [1996]. On the other hand, the weight of the empirical evidence in most of the countries studied to date indicates that interest rate volatility does vary by the level of the corresponding forward rate. To illustrate that fact, we studied the shortest forward rate on the U.S. Treasury curve on a daily basis from January 2, 1962 through December 31, 2020. We ordered the data from lowest forward rate level to highest forward rate level. We formed non-overlapping groups using the larger of 50 observations or 1/100th of total observations each and calculated both the standard deviation of 91-day forward rate changes and the mean beginning-of-period forward rate in each group. The results are plotted in Exhibit III:

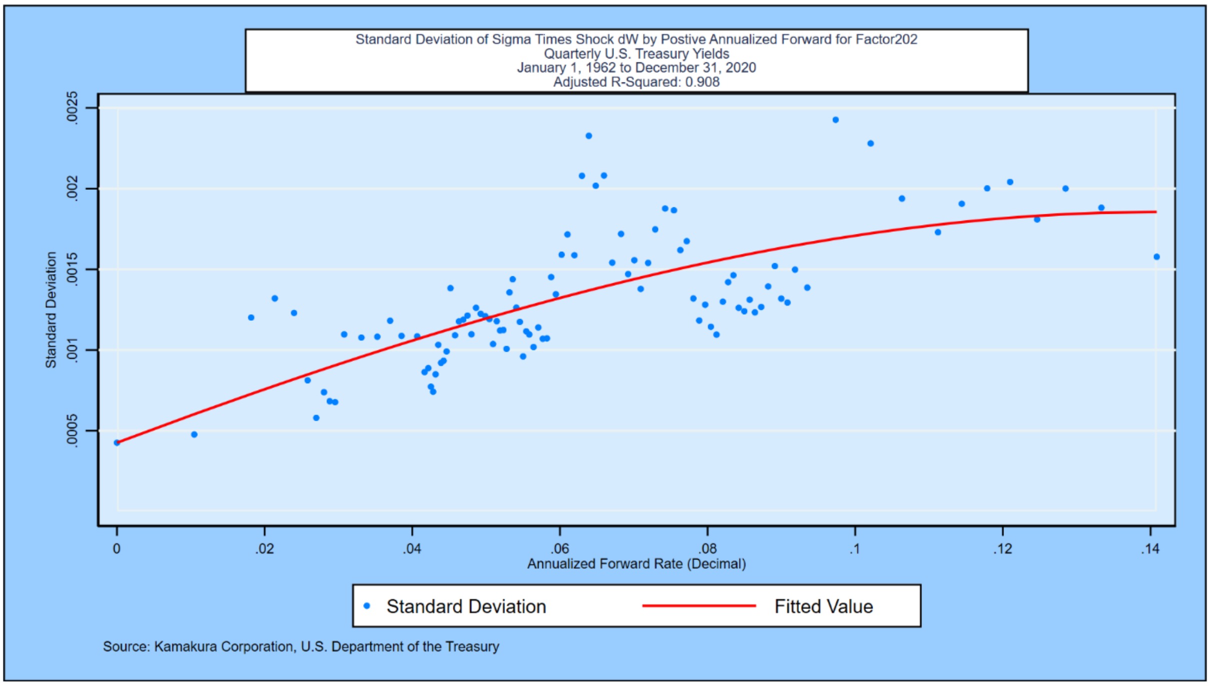
Exhibit III



A cubic function of annualized forward rates explains 92.4% of the variation in the standard deviation of forward rate changes for these ordered groups. This is the volatility function used when extracting the first random factor driving the U.S. Treasury curve. Note that the right-hand side of the curve has been constrained to have a first derivative of zero at a high level of rates.[2] The rise in volatility in higher rate environments has been confirmed in the government securities markets for Australia, Canada, France, Germany, Italy, Japan, Russia, Singapore, Spain, Sweden, Thailand, the United Kingdom, and the United States. Exhibit IV shows the results for the second risk factor in the U.S. Treasury market, the idiosyncratic movements in the quarterly forward rate maturing in 10 years:

[2] This constraint is one method for imposing the cap in stochastic volatilities suggested by Heath, Jarrow and Morton [Econometrica, 1992] to prevent a positive possibility of (a) infinitely high rates or (more practically) (b) unrealistically high rates.

Exhibit IV



The cubic stochastic volatility specification explains 90.8% of the observed variation in forward rate volatility in the quarterly forward rate maturing at the 10-year point on the U.S. Treasury yield curve. We have imposed the same constraint on the first derivative and require that the fitted volatility not be less than the observed volatility when interest rates are negative, which we discuss later in this section.

Exhibit V shows the historical movements in U.S. Treasury zero-coupon yields over the historical period studied:

Exhibit V

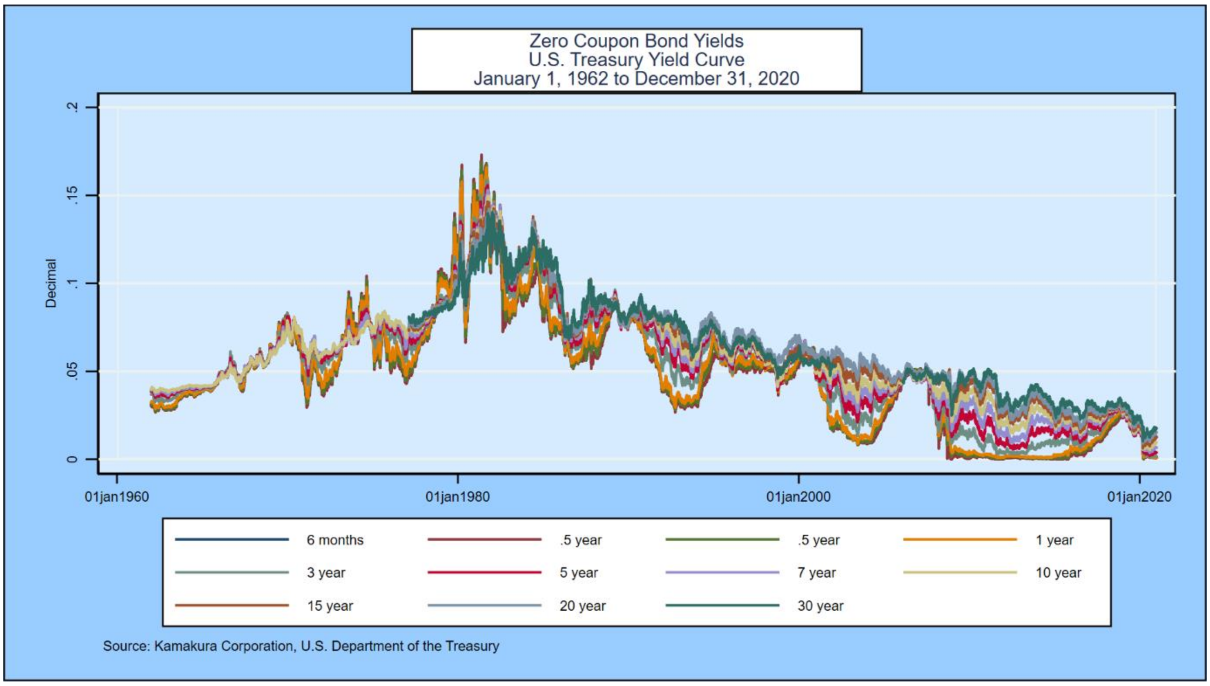
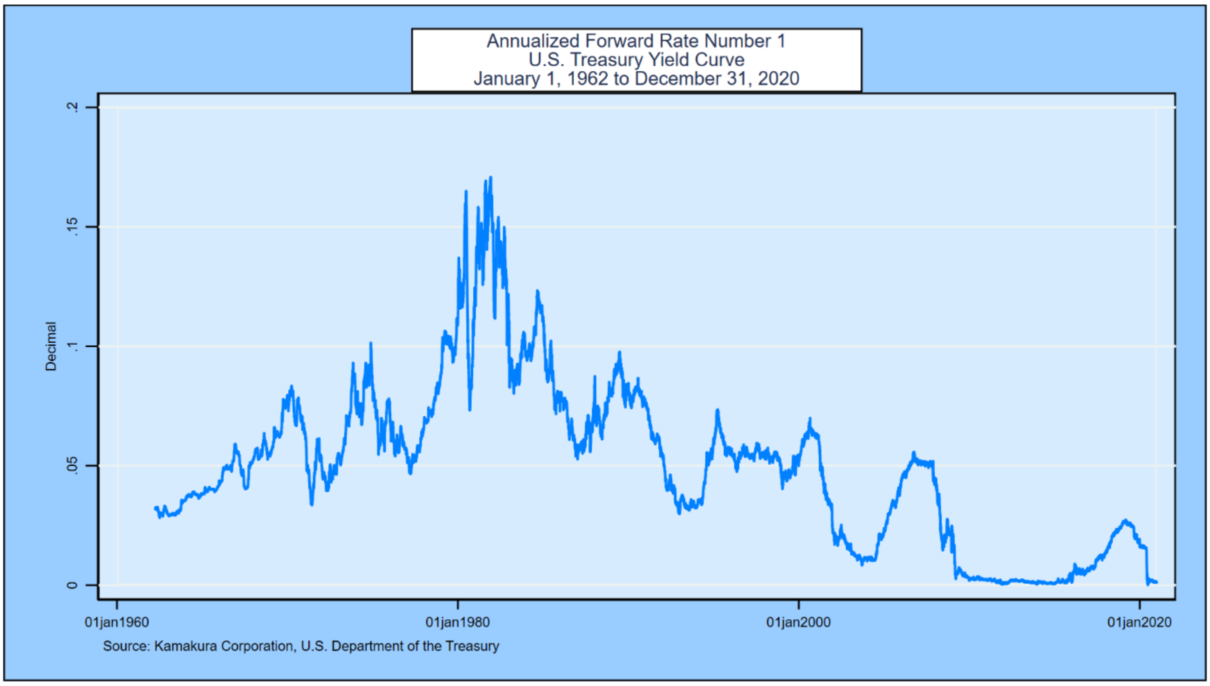


Exhibit VI below shows the evolution of the first quarterly forward rate (the forward that applies from the 91st day through the 182nd day) over the same time period:

Exhibit VI



We use three statistical tests to determine whether or not the hypothesis of normality for forward rates and zero-coupon bond yields should be rejected at the 5% level: the Shapiro-Wilk test, the Shapiro-Francia test, and the skew test, all of which are available in common statistical packages. The U.S. data history is now larger than the maximum number of observations suggested by these tests. In early work, when the sample size was smaller, the hypothesis of normality was rejected at each 91-day forward rate maturity date. We re-examine this hypothesis as needed using smaller samples from the full data set on a regular basis.

The rejection of the hypothesis of normality is problematic for constant coefficient or "affine" term structure models. In most of the other countries studied, the hypothesis of normality has been rejected strongly as well. Given these results, we proceed with caution on the implementation of the affine model.

In Chapter 3 of Advanced Financial Management (second edition, 2013), van Deventer, Imai and Mesler analyze the frequency with which U.S. Treasury forward rates move up together, down together or remain unchanged. This exercise informs the Heath, Jarrow and Morton parameter fitting process and is helpful for the model validation questions posed in the Appendix. We perform the yield curve shift analysis using 14,737 days of quarterly forward rates for the U.S. Treasury yield curve. We analyze the daily shifts in the forward rates on each business day from January 2, 1962 through December 31, 2020. The results are given in Table II:

Table II

Kamakura Corporation
HJM 10 Factor Model
U.S. Treasury
Using Daily Data from January 1, 1962 through December 31, 2020
Date of Analysis: December 31, 2020

Type of Yield Shift	Number of Observations	Percent of Observations
All yields shift up	1,562	10.60
All yields shift down	844	5.73
All yields are unchanged	3	0.02
Yield curve twists	12,328	83.65
Total	14,737	100.00

Kamakura Corporation, U.S. Department of the Treasury

Yield curve shifts were all positive, all negative, or all zero 10.60%, 5.73%, and 0.02% of the time, a total of 16.35% of all business days. The predominant yield curve shift was a twist, with a mix of positive changes, negative changes, or zero changes. These figures are similar to those for the 12 other countries for which we estimate term structure model parameters on a regular basis. These twists, which happen 83.65% of the time in the U.S., cannot be modeled accurately with the conventional implementation of one factor term structure models.

Another important aspect of yield curves is the number of local minima and maxima that have occurred over the modeling period. The results for the U.S. Treasury Market are given in Table III:

Table III

Analysis of Number of Local Minima and Maxima Each Day		
Number of Humps	**Number of Observations**	**Percent of Observations**
0 local minimum and maximum	4,702	31.91
1	3,357	22.78
2	3,175	21.54
3	1,388	9.42
4	807	5.48
5	771	5.23
6	244	1.66
7	281	1.91
8	12	0.08
9	0	0.00
10 or more	0	0.00

Kamakura Corporation, U.S. Department of the Treasury

The number of days with 0 or 1 humps (defined as the sum of local minima and maxima on that day's yield curve) was 54.69% of the total observations in the data set.

Finally, before proceeding, we count the number of occurrences of negative rates for each forward rate segment of the yield curve over the history provided by the U.S. Department of the Treasury and report on the observed 91-day volatility of forward rates when the start of the period annualized forward rate is negative, zero, and positive.

Table V

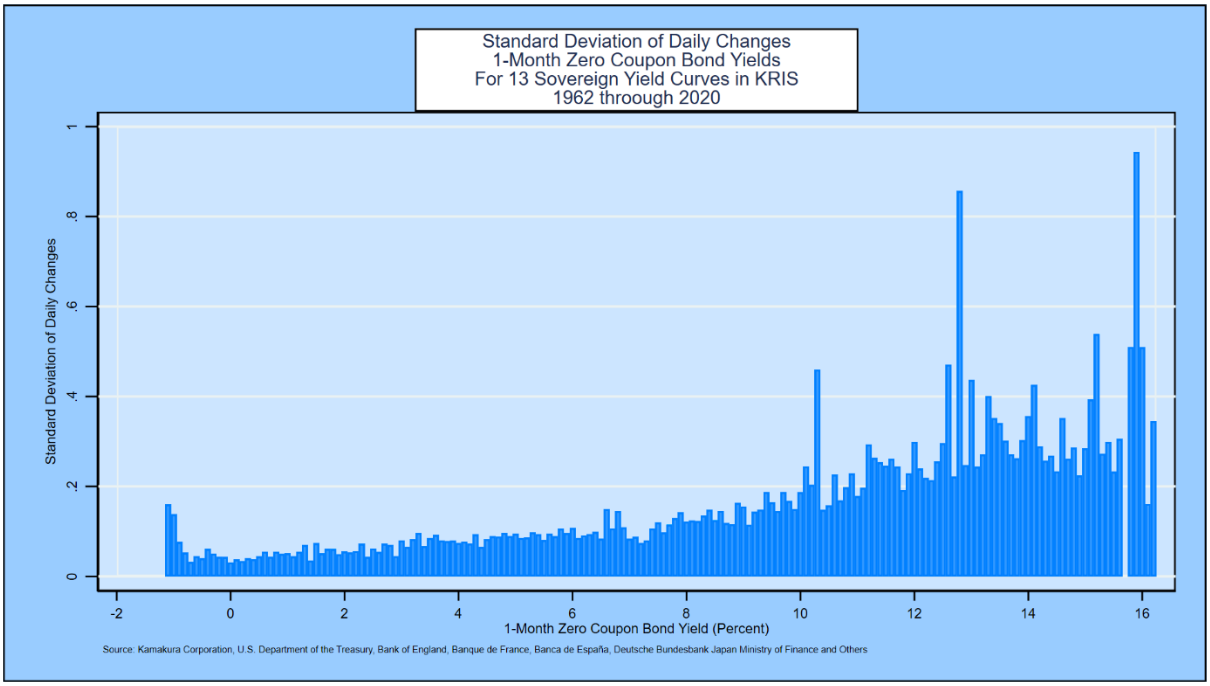
Kamakura Corporation				
HJM 10 Factor Model				
U.S. Treasury				
Using Daily Data from January 1, 1962 through December 31, 2020				
Date of Analysis: December 31, 2020				
Source: Kamakura Corporation, U.S. Department of the Treasury				
Count of Negative Quarterly Forward Rates				
Quarter Number	**Observations**	**Negative**	**Zero**	**Positive**
1	14737	0	0	14737
2	14737	0	0	14737
3	14737	0	0	14737
4	14737	0	0	14737
5	14737	0	0	14737
6	14737	0	0	14737
7	14737	0	0	14737
8	14737	0	0	14737
9	14737	0	0	14737
10	14737	0	0	14737
11	14737	0	0	14737
12	14737	0	0	14737
13	14737	0	0	14737
14	14737	0	0	14737
15	14737	0	0	14737
16	14737	0	0	14737
17	14737	0	0	14737
18	14737	0	0	14737
19	14737	0	0	14737
20	14737	0	0	14737
21	14737	0	0	14737
22	14737	0	0	14737
23	14737	0	0	14737
24	14737	0	0	14737

The U.S. Treasury, in part because of Department policy outlined on its website, has not reported any observations for which quarterly forward rates have been negative.

The same table for Japan shows that the volatility of forward rate changes can be calculated for the first forward rate on 303 observation dates when that forward rate was negative. The 91-day volatility was 0.018553%. For the 10,425 observation dates for which the first forward rate was positive, the volatility over 91 days was 0.135174%. For other forward rate maturities, the volatility of the negative rate observations gradually increased with maturity. We emphasize two obvious points: rates can be and have been negative, and, when rates hit zero and below, interest rate volatility is not zero. It is positive but at a lower level than for positive forward rate observations.

Observed volatility in 1-month zero-coupon bond yields as a function of interest rate level for the standard 13-country "World" data set is shown in Exhibit VII:

Exhibit VII



III. Fitting Heath, Jarrow and Morton Parameters

A simple first step in constructing a multi-factor Heath, Jarrow and Morton model is to conduct principal components analysis on the forward rates that make up the relevant yield curve. For the U.S. Treasury market, at its longest maturity, these quarterly segments consist of one three-month spot rate and 119 forward rates. Over 9,845 observations, the principal components analysis indicates in Table V that the first factor explains only 53.87% of the movement in forward rates over the full curve. For a high degree of explanatory power, the principal components analysis indicates that 8 to 12 factors will be necessary.

Table V

```
Principal components/correlation                 Number of obs    =      9,845
                                                 Number of comp.  =         16
                                                 Trace            =        119
    Rotation: (unrotated = principal)            Rho              =     1.0000
```

Component	Eigenvalue	Difference	Proportion	Cumulative
Comp1	64.1059	28.9925	0.5387	0.5387
Comp2	35.1134	26.4621	0.2951	0.8338
Comp3	8.65128	3.05648	0.0727	0.9065
Comp4	5.5948	2.72957	0.0470	0.9535
Comp5	2.86522	1.59842	0.0241	0.9776
Comp6	1.2668	.525613	0.0106	0.9882
Comp7	.741188	.380692	0.0062	0.9944
Comp8	.360495	.201079	0.0030	0.9975
Comp9	.159416	.0766308	0.0013	0.9988
Comp10	.0827856	.038192	0.0007	0.9995
Comp11	.0445936	.0343191	0.0004	0.9999
Comp12	.0102746	.00739583	0.0001	1.0000
Comp13	.00287874	.00205533	0.0000	1.0000
Comp14	.000823409	.000746749	0.0000	1.0000
Comp15	.0000766598	.0000525294	0.0000	1.0000
Comp16	.0000241304	.0000216175	0.0000	1.0000
Comp17	2.51287e-06	1.55680e-06	0.0000	1.0000
Comp18	9.56066e-07	9.56066e-07	0.0000	1.0000
Comp19	0	0	0.0000	1.0000

With this analysis as background, we begin the Heath, Jarrow and Morton fitting process.

In the studies done so far, the number of statistically significant factors are summarized below:

Australia:	Commonwealth Government Securities, 1996	11 factors
Canada:	Government of Canada Securities, 2005	11 factors
France:	French Government Bonds, 2015	7 factors
Italy:	Italian Government Bonds, 2015	11 factors
Germany:	Bunds, 1997	15 factors
Japan:	Japanese Government Bonds, 1974	10 factors
Russia:	Russian Government Bonds, 2003	11 factors
Singapore:	Singapore Government Securities, 1998	10 factors
Spain:	Spanish Government Securities, 1987	11 factors
Sweden:	Swedish Government Securities, 1987	11 factors
Thailand:	Thai Government Securities, 1999	11 factors
United Kingdom:	Government Securities, 1979	15 factors
United States:	Treasury Securities, prior version, 1962	10 factors
World:	Government Securities in 13 Countries, 1962	12 factors

Note that our prior term structure model fitting exercise for the U.S. Treasury market resulted in 10 statistically significant factors through December 31, 2019.

We now fit a multi-factor Heath, Jarrow and Morton model to U.S. Treasury zero-coupon yield data from January 2, 1962 to December 31, 2020. For computational simplicity, we compress the data regimes numbered in the right-hand column of Table I to two regimes. The first is for observations where no maturity longer than 10 years was reported. The second is for those observations where no maturity longer than 30 years was reported.

The procedures used to derive the parameters of a Heath, Jarrow and Morton model are described in detail in Jarrow and van Deventer (June 16, 2015 and May 5, 2017).

We followed these steps to estimate the parameters of the model:

- We extract the zero-coupon yields and zero-coupon bond prices for all quarterly maturities out to 30 years for all daily observations for which the 30-year zero-coupon yield is available. For other observations, we extended the analysis to the longest maturity available, which varies by data regime. This is done using Kamakura Risk Manager, version 10.1, using the maximum smoothness forward rate approach to fill the quarterly maturity gaps in the zero-coupon bond data.
- We use overlapping 91-day intervals to measure changes in forward rates, avoiding the use of "quarterly" data because of the unequal lengths of calendar quarters. Because overlapping observations trigger auto-correlation, "HAC" (heteroscedasticity and autocorrelation consistent) standard errors are used. The methodology is that of Newey-West with 91-day lags.
- We consider ten potential explanatory factors: the idiosyncratic portion of the movements in quarterly forward rates that mature in 6 months, 1 year, 1.5, 2, 3, 5, 7, 10, 20, and 30 years. Ten factors are required by the Bank for International Settlements market risk guidelines published in January 2016 and relevant to the Fundamental Review of the Trading Book.
- We calculate the discrete changes in forward returns as described in the parameter technical guide. Because the discrete changes are non-linear in the no-arbitrage framework of Heath, Jarrow and Morton, we use non-linear least squares to fit interest rate volatility.
- We use a different non-linear regression for each segment of the yield curve. We considered a panel-based approach, but we rejected it for two reasons: first, the movement of parameters as maturity lengthens is complex and not easily predictable before estimation; second, the residual unexplained error in forward rates is very, very small, so the incremental merit of the panel approach is minimal.
- We then begin the process of creating the orthogonalized risk factors that drive interest rates using the Gram-Schmidt procedure. These factors are assumed to be uncorrelated independent random variables that have a normal distribution with mean zero and standard deviation of 1.
- Because interest volatility is assumed to be stochastic, simulated out-of-sample forward rates will not in general be normally distributed. We also calculate constant volatility parameters and choose the most accurate from the constant volatility and stochastic volatility models estimated.
- In the estimation process, we added factors to the model as long as each new factor provided incremental explanatory power. The standard suite of

models in both cases includes 1 factor, 2 factors, 3 factors, 6 factors and "all factors," which varies by country.

We postulate that interest rate volatility for each forward rate maturity k is a cubic function of the annualized forward rate that prevails for the relevant risk factor j at the beginning of each 91-day period:

$$\sigma_{jk} = \max\,[b_{0,jk}, b_{0,jk} + b_{1,jk}f + b_{2,jk}f^2 + b_{3,jk}f^3]\ if\ f > 0,$$
$$\sigma_{jk} = b_{0,jk}\ if\ f \leq 0,$$

When the initial forward rate is negative, we postulate that interest rate volatility is a constant. Using Japan volatility data reported above, the measured $b_{0,jk}$ was 0.018553%. Using the 13-country World model, the value of $b_{0,jk}$ was 0.03483%.

We use the resulting parameters and accuracy tests to address the hypothesis that a one factor model is "good enough" for modeling U.S. Treasury yields in the Appendix. We report the accuracy results for 1, 2, 3, 6 and all (10) factors. The factors are the idiosyncratic variation in quarterly forward rates at each of 10 maturities. The factors, described by the maturity of the quarterly forward rate used, are added to the model in this order:

Data Regime 1 (Maturities of 10 years or less)

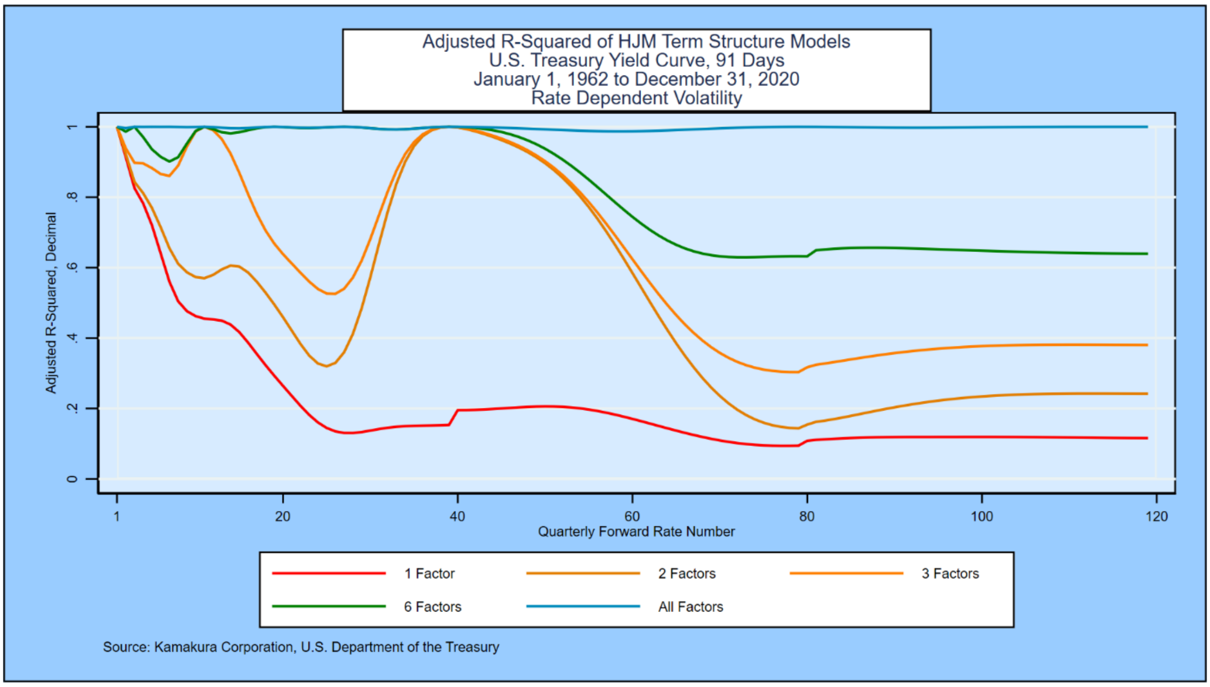
Factor 1: 6 months
Factor 2: 10 years
Factor 3: 3 years
Factor 4: 7 years
Factor 5: 1 year
Factor 6: 5 years
Factor 7: 2 years
Factor 8: 1.5 years

Data Regime 2 (Maturities longer than 10 years)

Factor 9: 30 years
Factor 10: 20 years

Exhibit VIII summarizes the adjusted r-squared for the non-linear equations for each of the 119 quarterly forward rate segments that make up the U.S. Treasury yield curve:

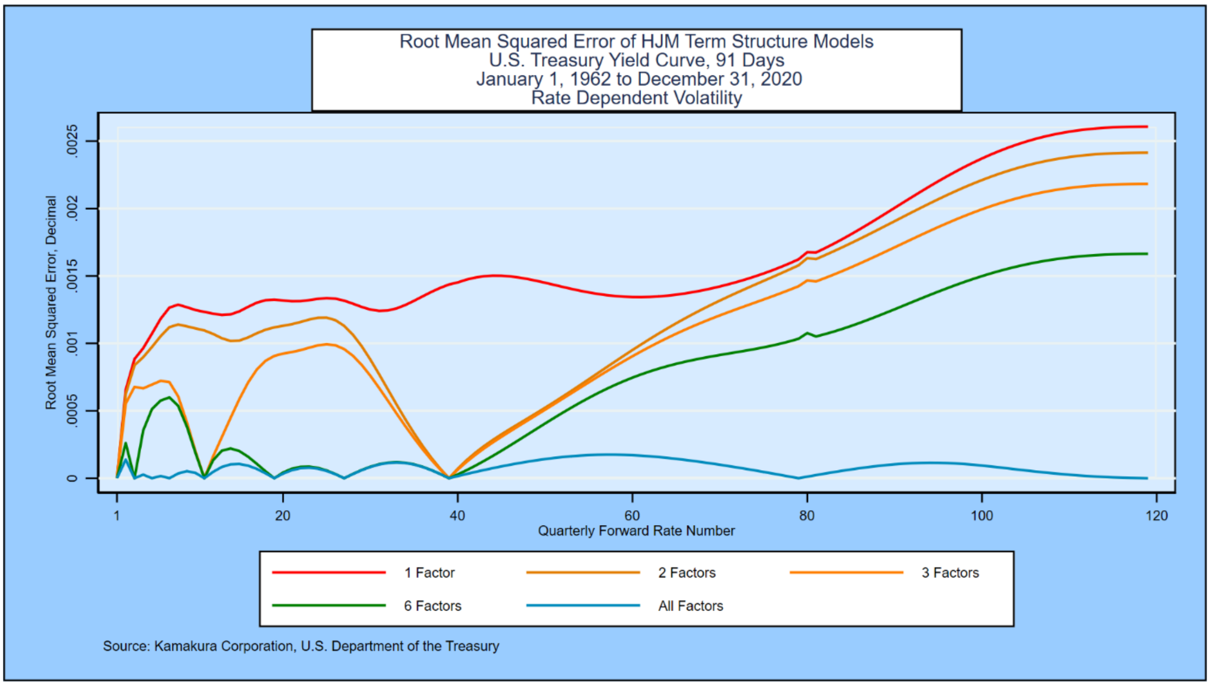
Exhibit VIII



The adjusted r-squared for the best practice model over each of the forward rates is plotted in blue and is near 100% for all 119 quarterly segments of the yield curve. The one factor model in red, by contrast, does a poor job of fitting 91-day movements in the quarterly forward rates. The adjusted r-squared is good, of course, for the first forward rate since the short rate is the standard risk factor in a one factor term structure model. Beyond the first quarter, however, explanatory power declines rapidly. The adjusted r-squared of the one factor model never exceeds 20% after the first 25 quarterly forward rates and is below that level at most maturities.

The root mean squared error for the 1, 2, 3, 6 and all (10) factor stochastic volatility model is shown in Exhibit IX.

Exhibit IX



The root mean squared error for the 10-factor model is less than 0.015% at every maturity along the yield curve. This result should not come as a surprise to a serious analyst, because it is very similar to the results of the best practice Heath, Jarrow and Morton term structure models for the other 12 government bond markets studied.

Bayesian Considerations in Model Validation

Kamakura term structure model validation is conducted in the spirit of Bayesian iterative model building as outlined by Gelman et al. This quote[3] from Gelman et al [2013] explains the Bayesian estimation process:

"The process of Bayesian data analysis can be idealized by dividing it into the following three steps:

1. Setting up a full probability model—a joint probability distribution for all observable and unobservable quantities in a problem. The model should be consistent with knowledge about the underlying scientific problem and the data collection process.

2. Conditioning on the observed data: calculating and interpreting the appropriate posterior distribution—the conditional probability distribution of the unobserved quantities of ultimate interest, given the observed data.

3. Evaluating the fit of the model and the implications of the resulting posterior distribution: how well does the model fit the data, are the substantive conclusions reasonable, and how sensitive are the results to the modeling

[3] Gelman et all [2013], page 3.

assumptions in step 1? In response, one can alter or expand the model and repeat the three steps."

Jarrow and van Deventer go on to explain that the iterative process described above by Gelman et al is especially important in fitting Heath, Jarrow and Morton parameters for the following reasons:

a. Negative interest rates have been observed in Japan, Hong Kong and many European countries, but many other countries, including the U.S., have yet to experience negative rates. In the U.S. case, the U.S. Department of the Treasury notes on its website that it overrides observed negative yields in the market with zero values.
b. The "knowledge about the underlying scientific problem" from the historical data available is as follows: (1) negative rates are possible, (2) they are much less likely to occur than positive rates, (3) interest rate volatility that results when rates are negative is of high interest but the historical data is either limited or non-existent, depending on the country, and (4) an international data set would best shed light on this and other HJM issues.

There are other issues relevant to estimation:

c. As noted by Heath, Jarrow and Morton [1992], stochastic volatility driven by interest rate levels must be capped to avoid a positive probability of infinitely high interest rates.
d. Subject to this cap, most market participants expect interest rate volatility to rise as rates rise and that the interest rate volatility that prevails when rates are negative represents the lowest level of volatility that would prevail. Historical experience with negative rates so far around the world makes it clear that interest rate volatility does not go to zero at any rate level.
e. Most market participants believe that the empirical drift in forward rates that occurs (i.e., the change in observed empirical interest rates in the case where all interest rate shocks are zero) varies by the level of interest rates. The stochastic volatility model described in this paper assumes that empirical drift is a cubic function of annualized forward rates.

To summarize, a model validation effort in the Bayesian spirit would address at least these issues:

- Tests of smoothness of simulated curves
- Tests to confirm existence of negative rates in selected circumstances in the simulation
- Comparison of simulated risk neutral and empirical yields
- Time series distribution of simulated risk neutral and empirical yields

We conduct an inspection of these issues with the aid of a forward-looking out-of-sample simulation of U.S. Treasury yields with the following specifications:

- Yield curve: U.S. Treasury yields

- Date of yields: March 5, 2021
- Number of scenarios: 500,000
- Simulation time horizon: 30 years
- Simulation periodicity: 91 days (quarterly)

A. Smoothness Validation

First, we select a random sample of 10 scenarios at each time step and visually examine them for smoothness. We can also use the discrete formula for smoothness given above to identify any outliers and examine the scenarios in question.

Exhibit X: 1 year

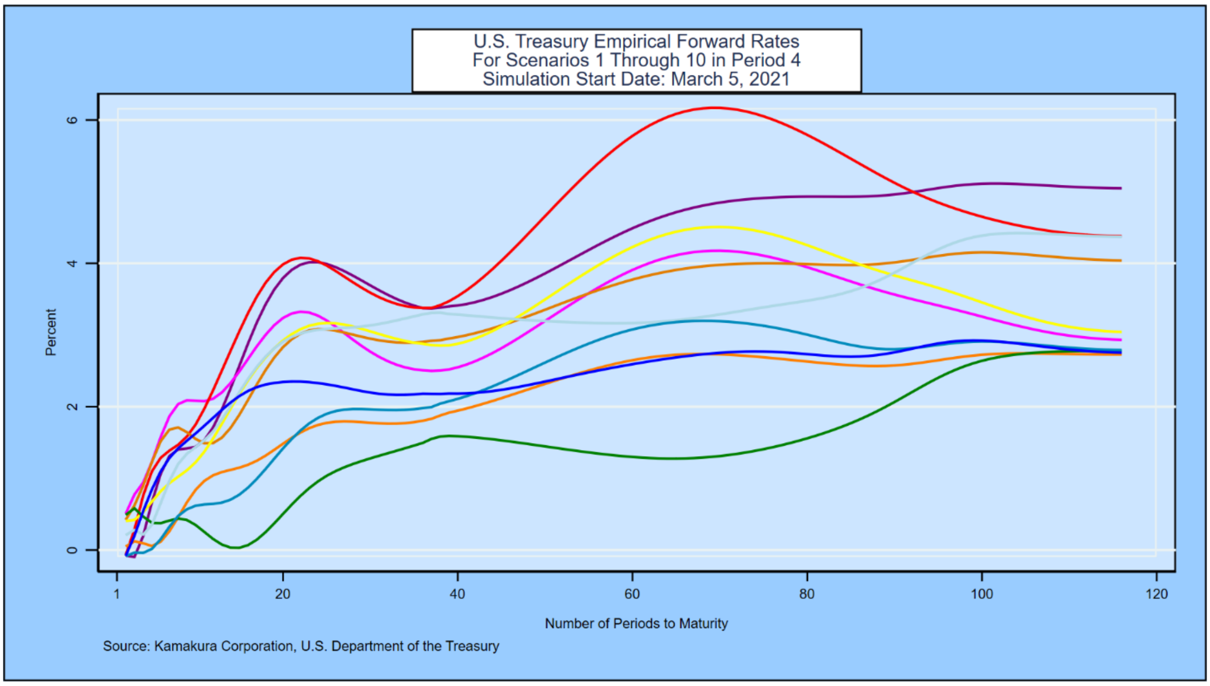


Exhibit XI: 5 years

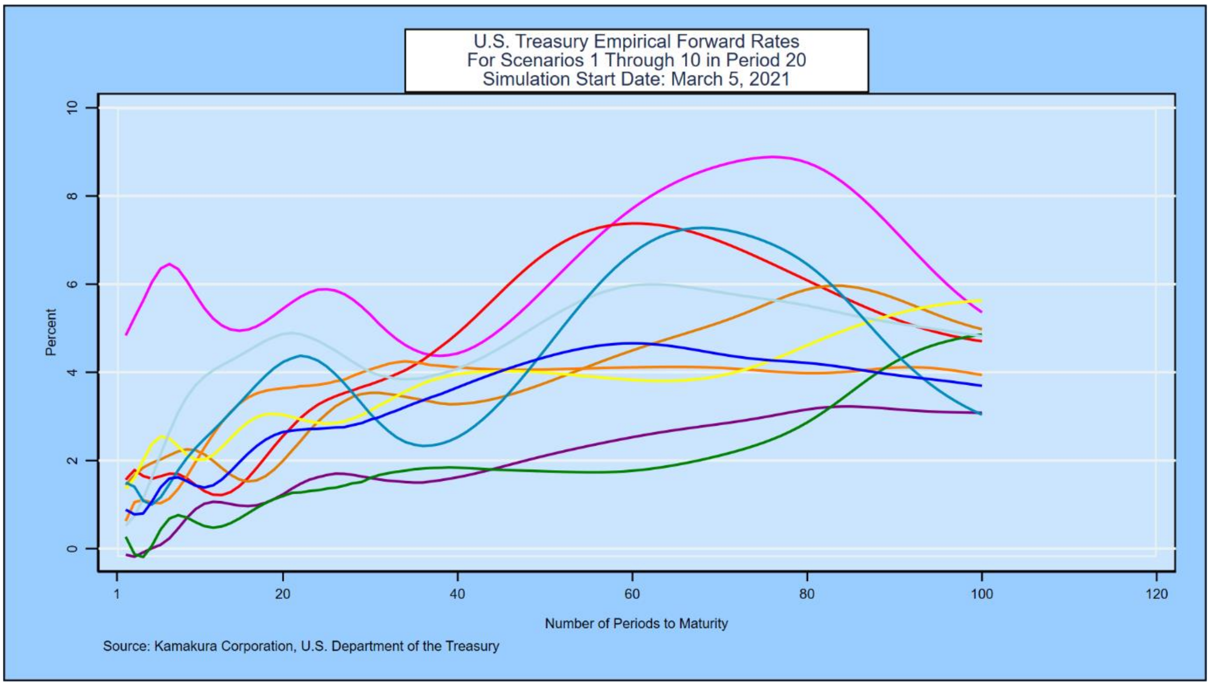


Exhibit XII: 10 years

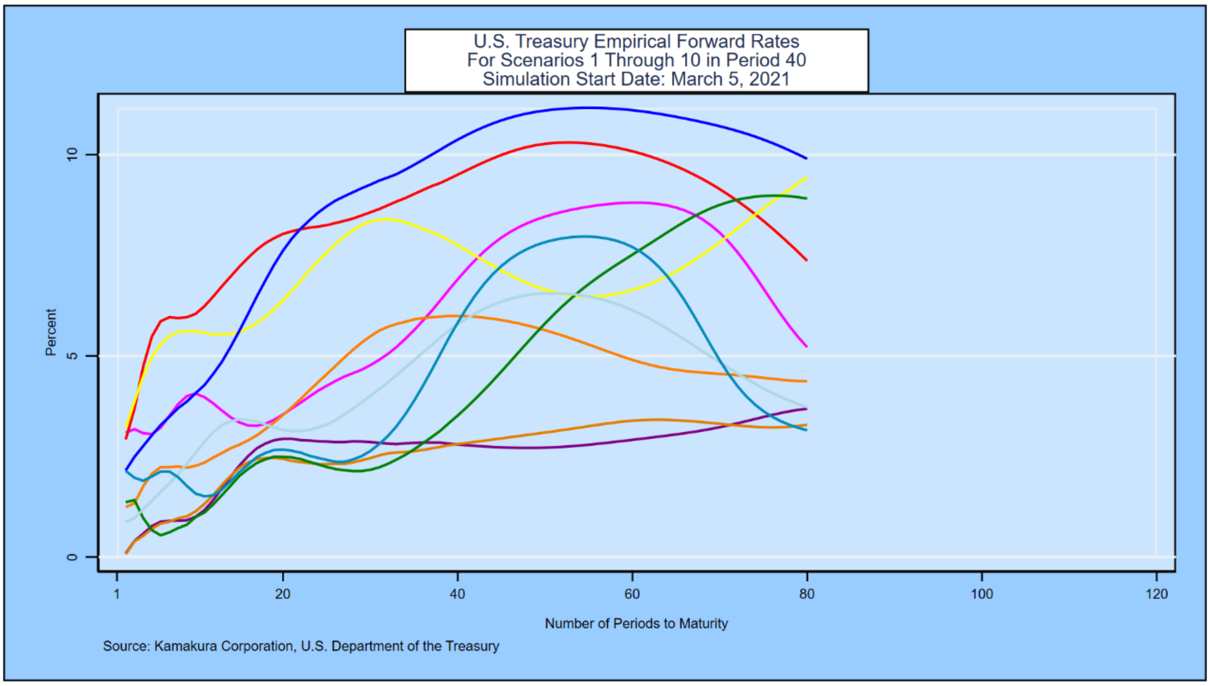


Exhibit XIII: 20 years

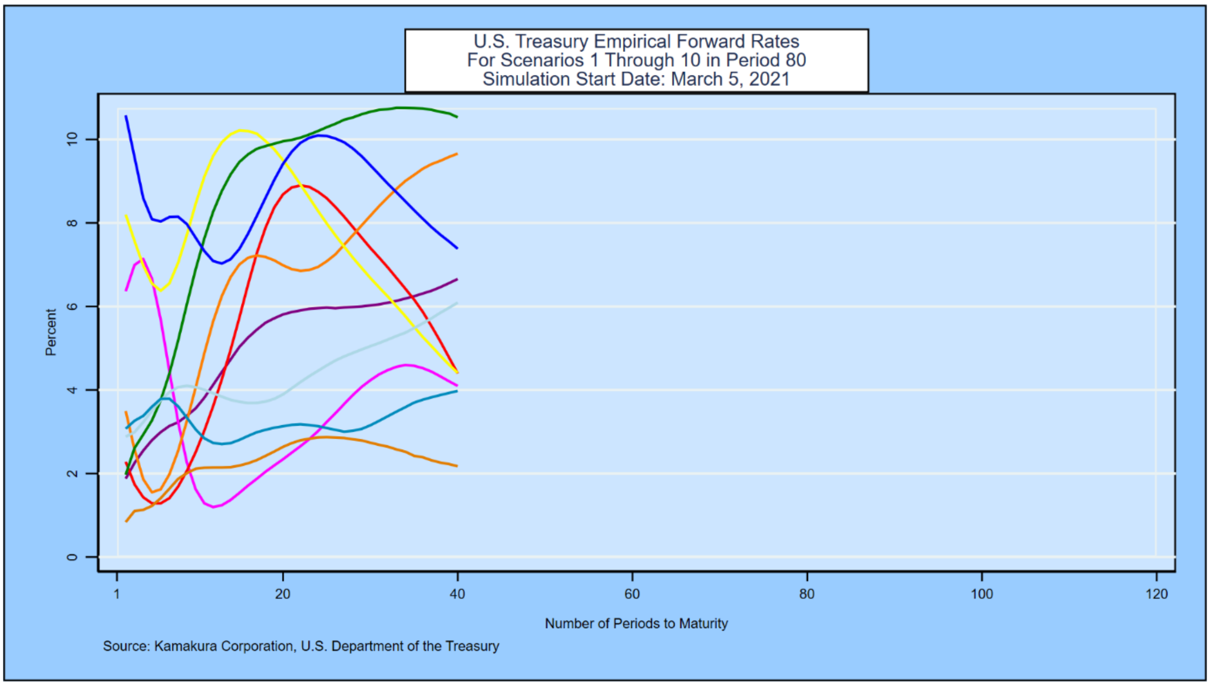
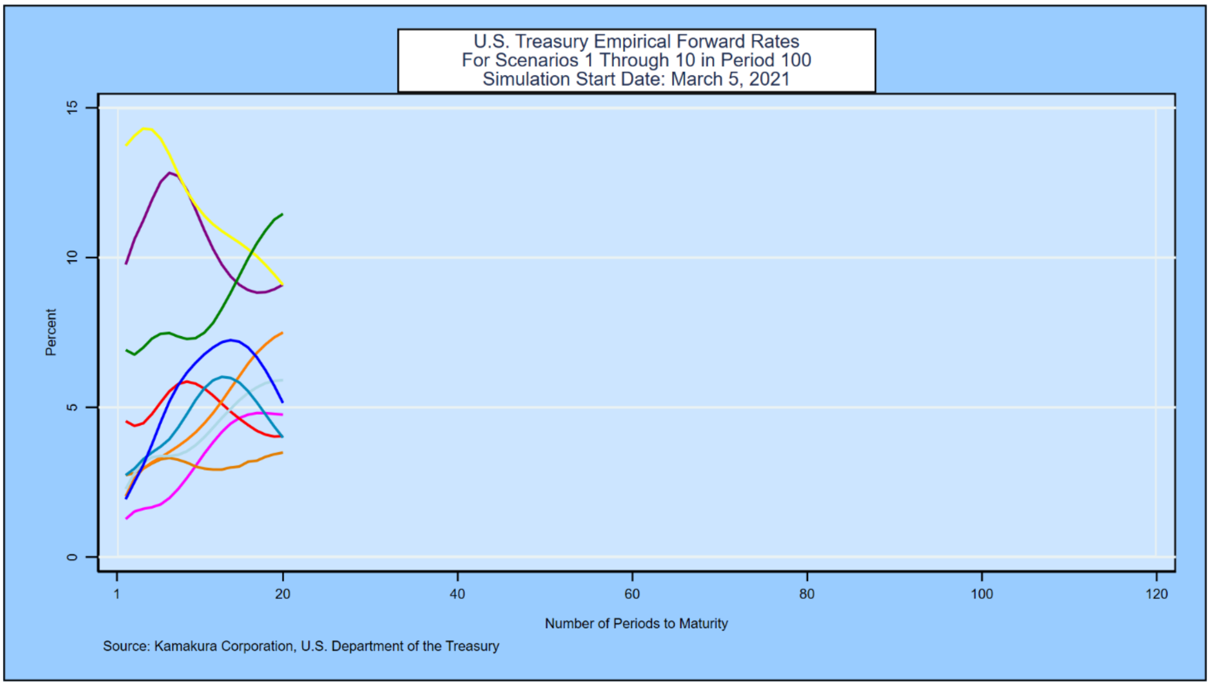


Exhibit XIV: 25 years



These graphs provide informal confirmation that nothing in the model estimation procedure has introduced artificial lumpiness in the simulated yield curves. A quantitative assessment of the smoothness of all 500,000 yield curves at each time step would provide the more formal confirmation that the yield curves simulated are realistically smooth.

B. Distribution of Simulated Risk Neutral and Empirical Rates

We now examine the probability distributions of risk neutral and empirical simulated rates at various maturities over time. We seek to detect visually any points in time at which the simulated distribution of yields is strange or unrealistic.

Exhibit XV: Three Month U.S. Treasury Yields at 1 Year

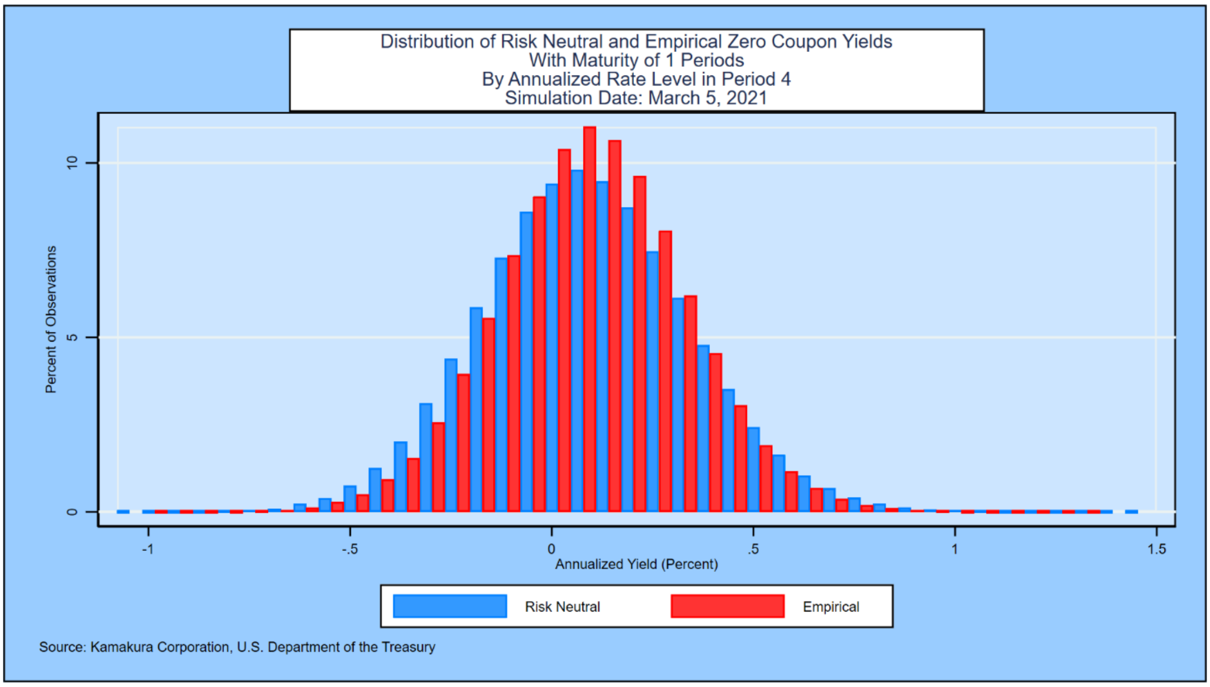


Exhibit XVI: Three Month U.S. Treasury Yields at 5 Years

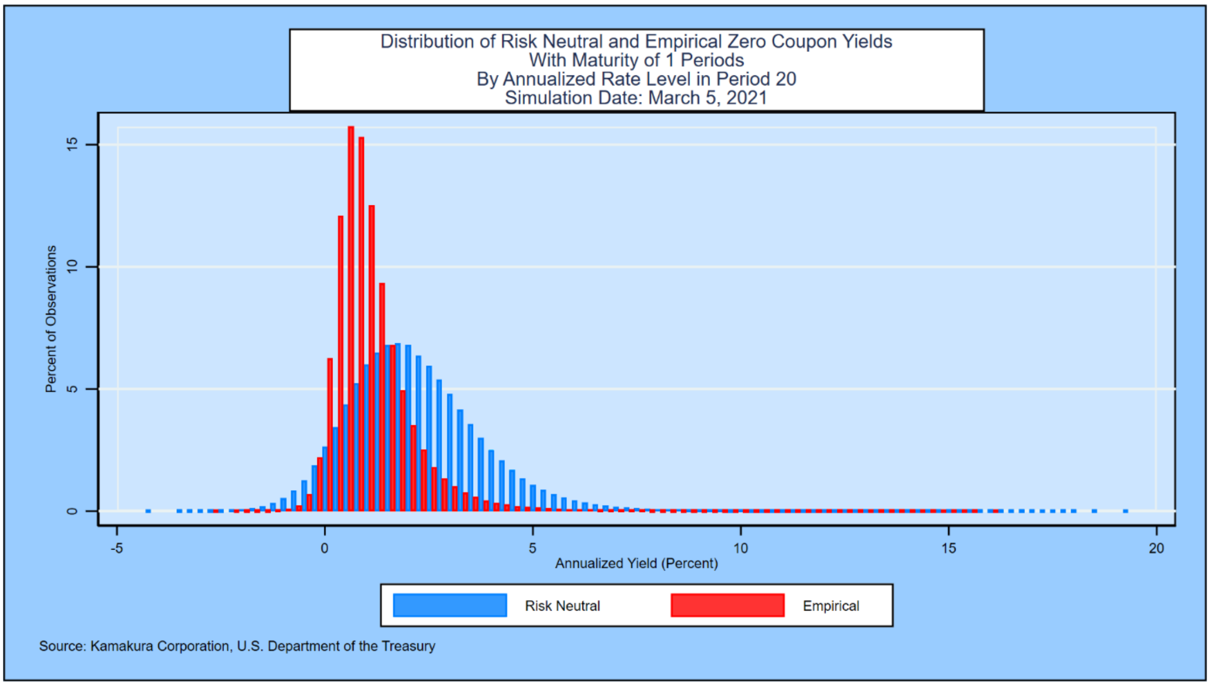


Exhibit XVII: Three Month U.S. Treasury Yields at 10 Years

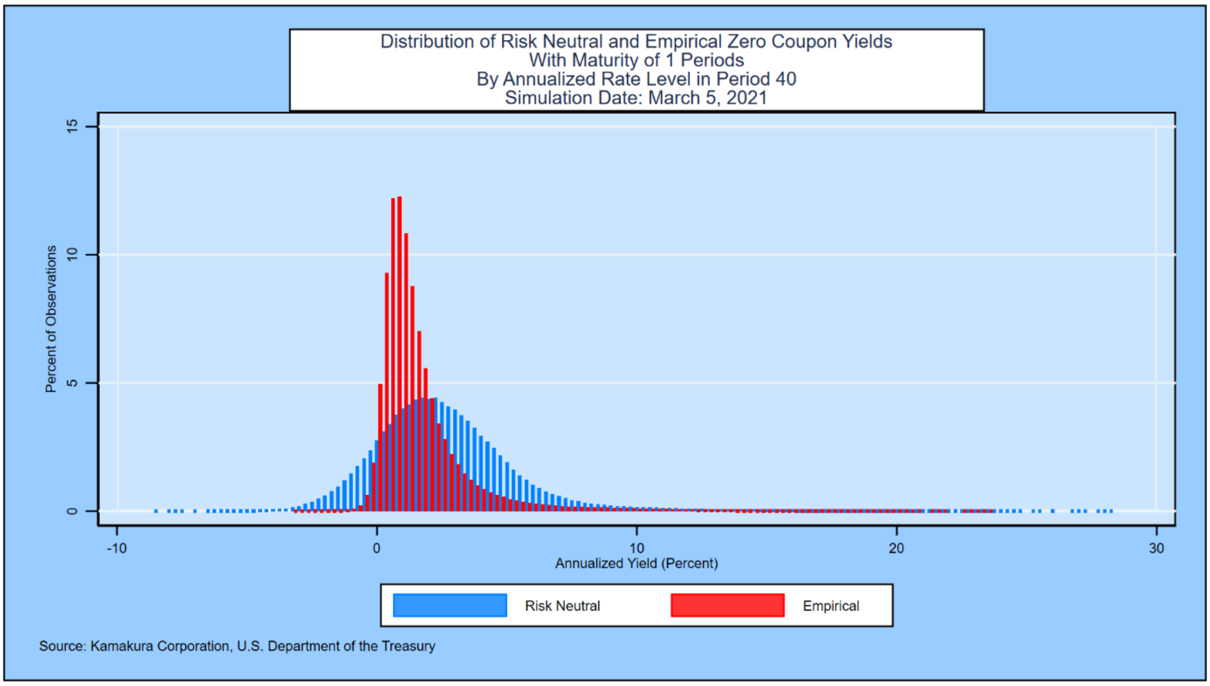


Exhibit XVIII: Three Month U.S. Treasury Yields at 20 Years

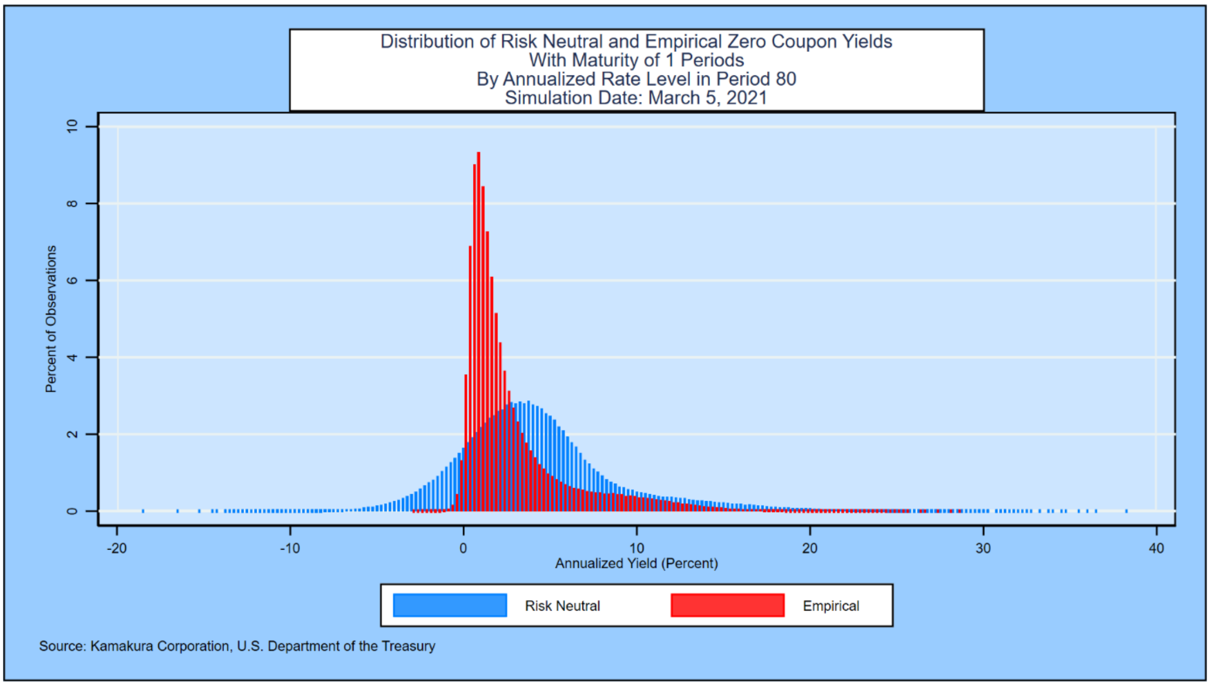


Exhibit XIX: Three Month U.S. Treasury Yields at 25 Years

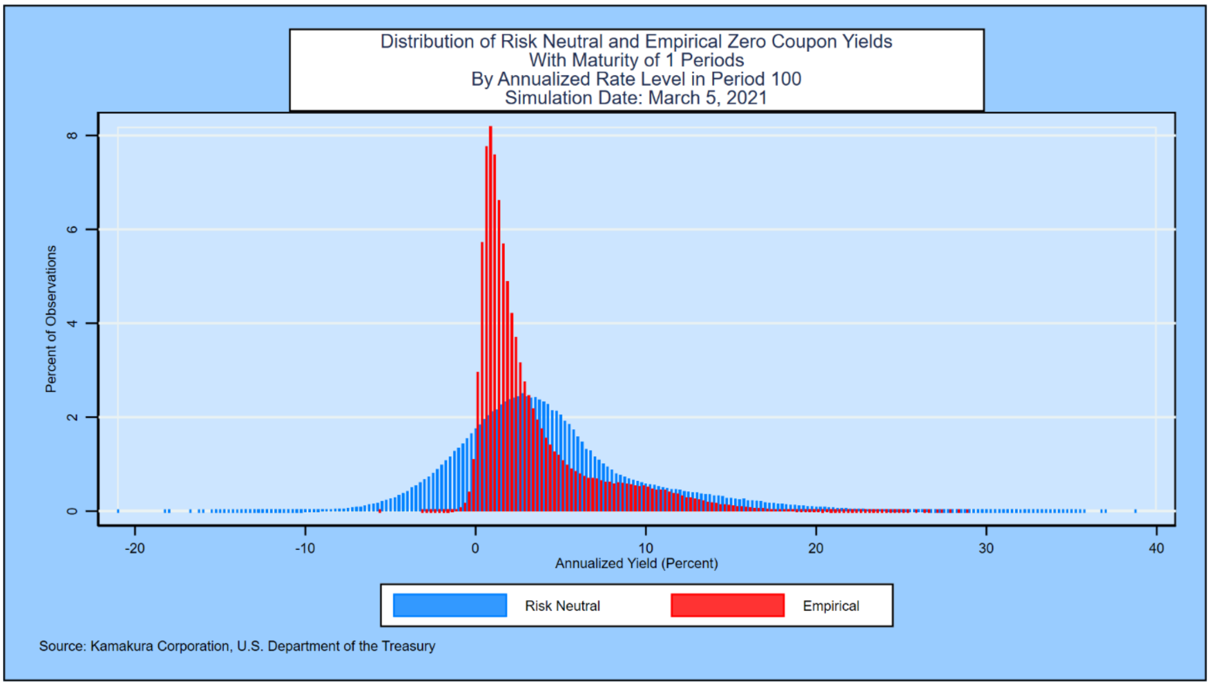


Exhibit XX: 10-year U.S. Treasury Yields at 1 Year

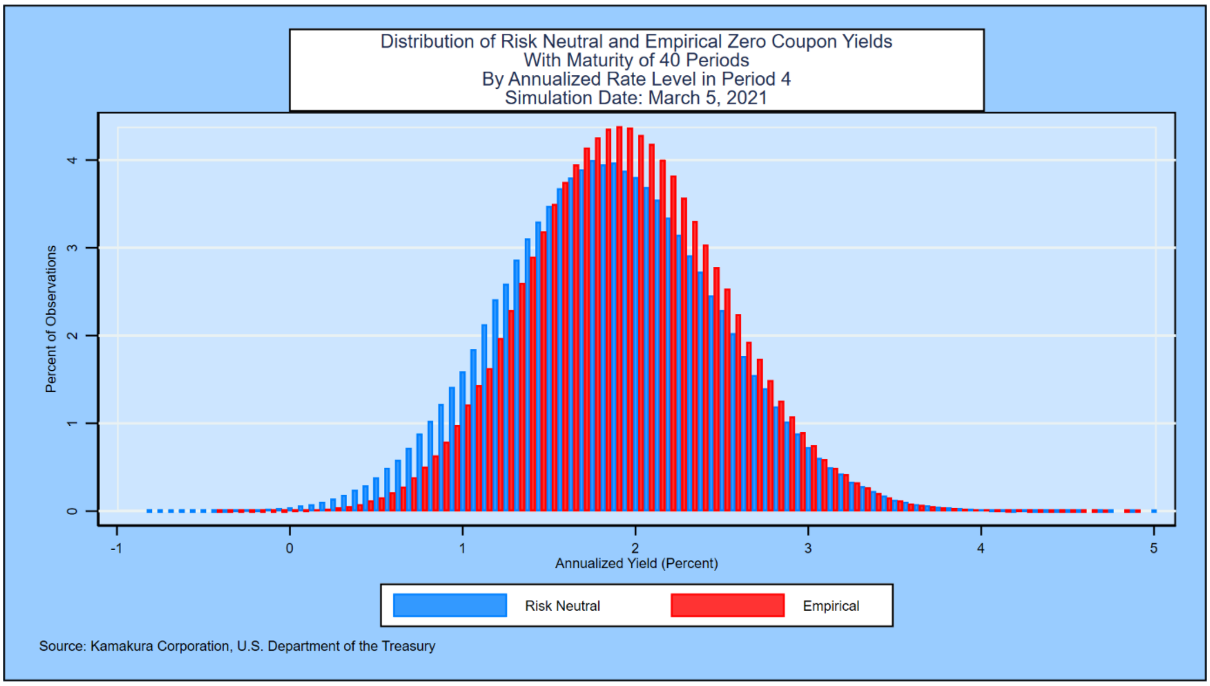


Exhibit XXI: 10-year U.S. Treasury Yields at 5 Years

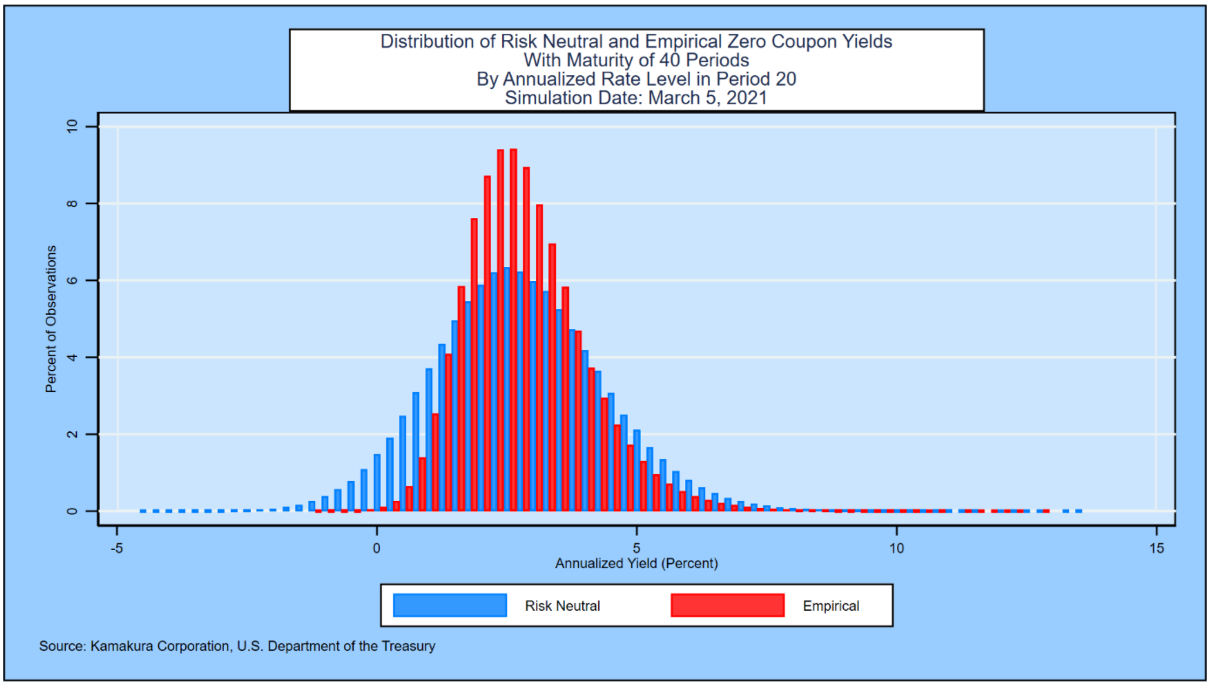


Exhibit XXII: 10-year U.S. Treasury Yields at 10 Years

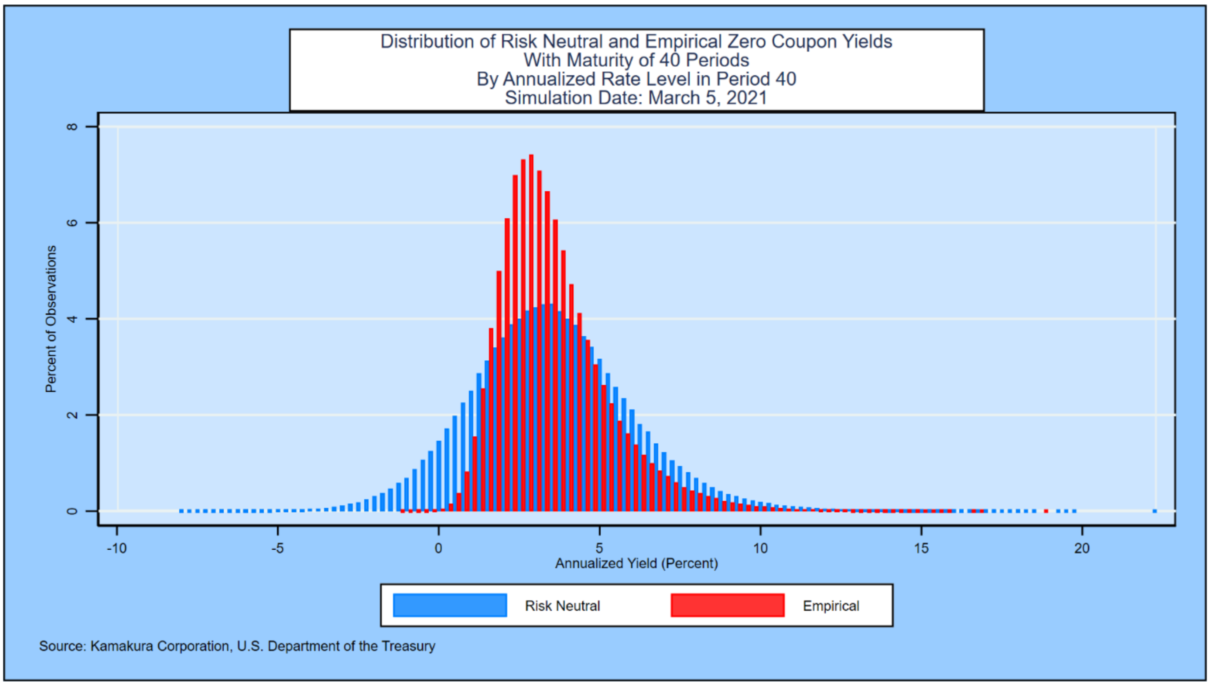
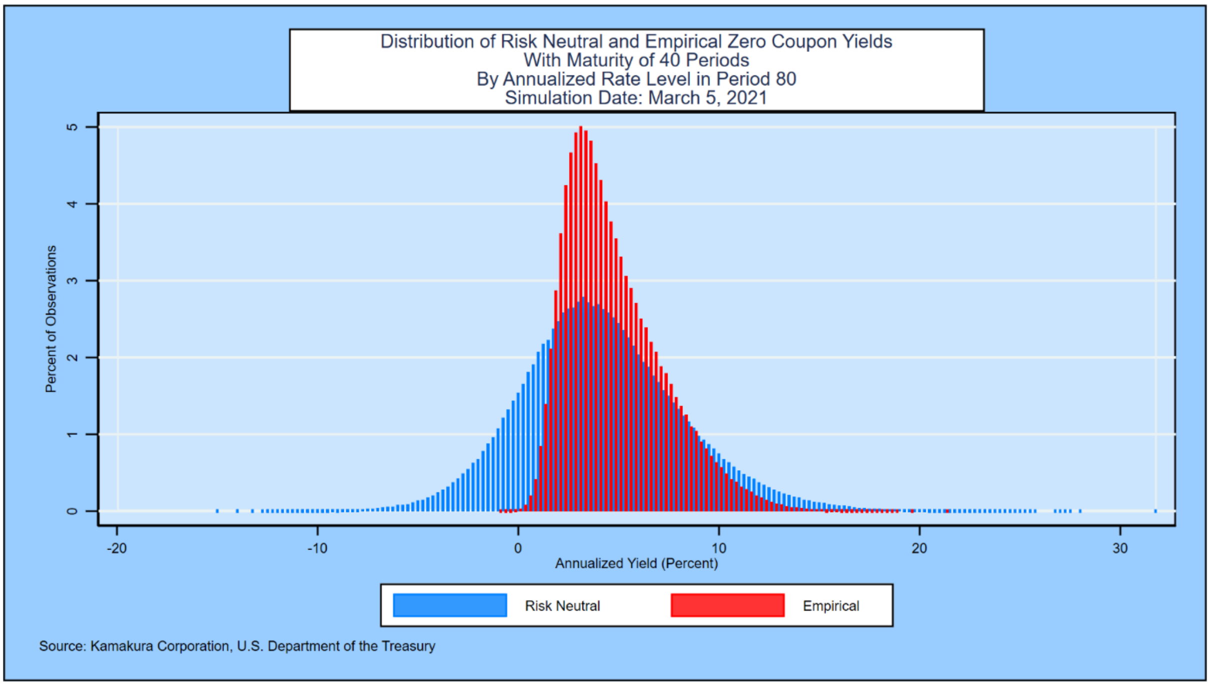


Exhibit XXIII: 10-year U.S. Treasury Yields at 20 Years



We conclude that the simulation is reasonable from multiple dimensions. Rates can be negative but (for empirical yields) the probability of negative rates is low. On the end of the spectrum, rates do rise to the 20% range but with a very low probability.

C. Time Series Distribution of Simulated Yields

We now plot the time series graphs of the mean, median, high, low and various percentiles for empirical rates.

Exhibit XXIV: 3-month Yields

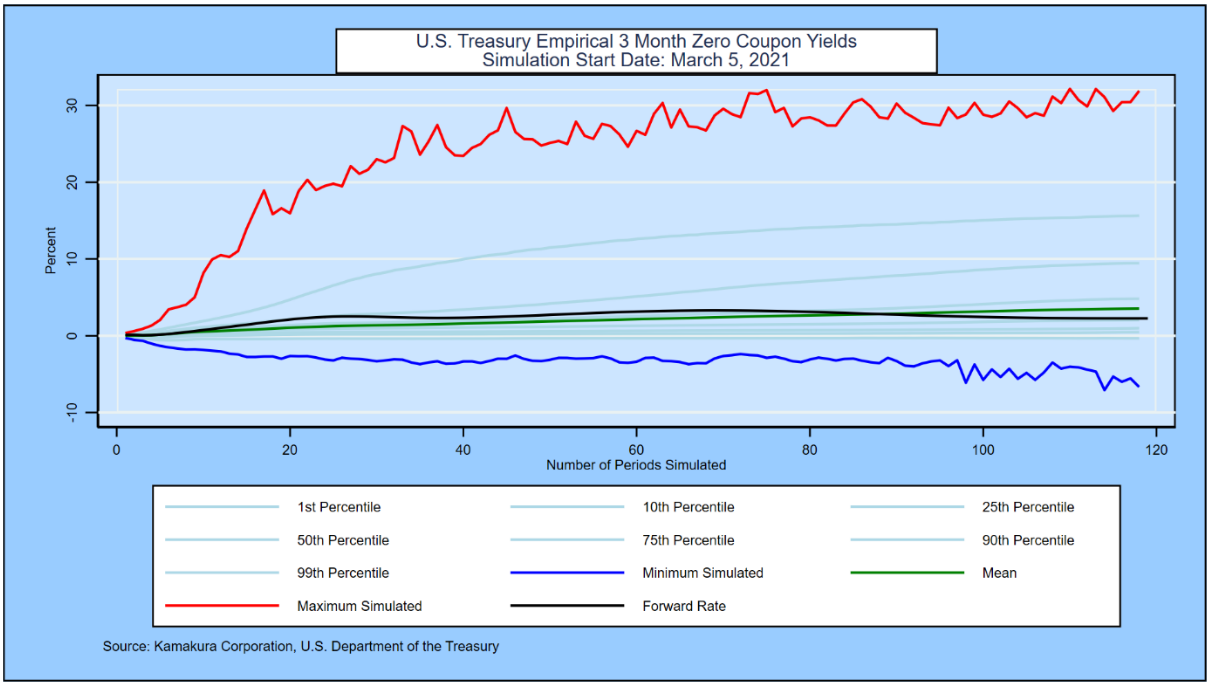


Exhibit XXV: 1 Year Yields

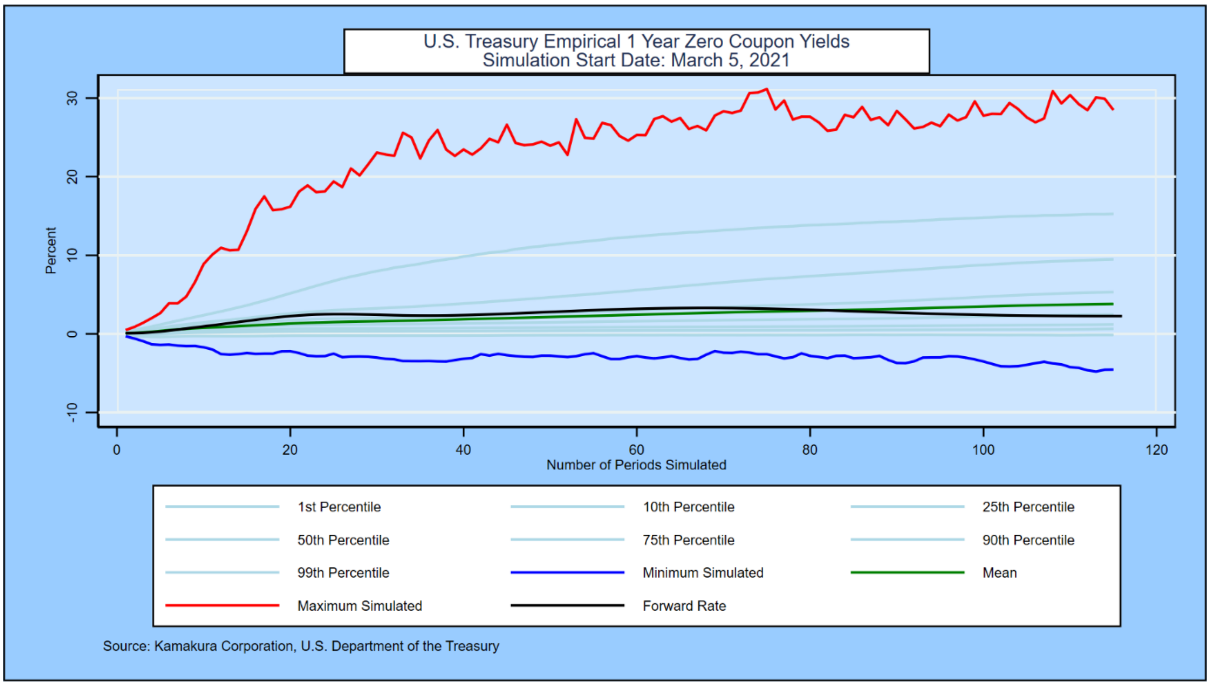


Exhibit XXVI: 5 Year Yields

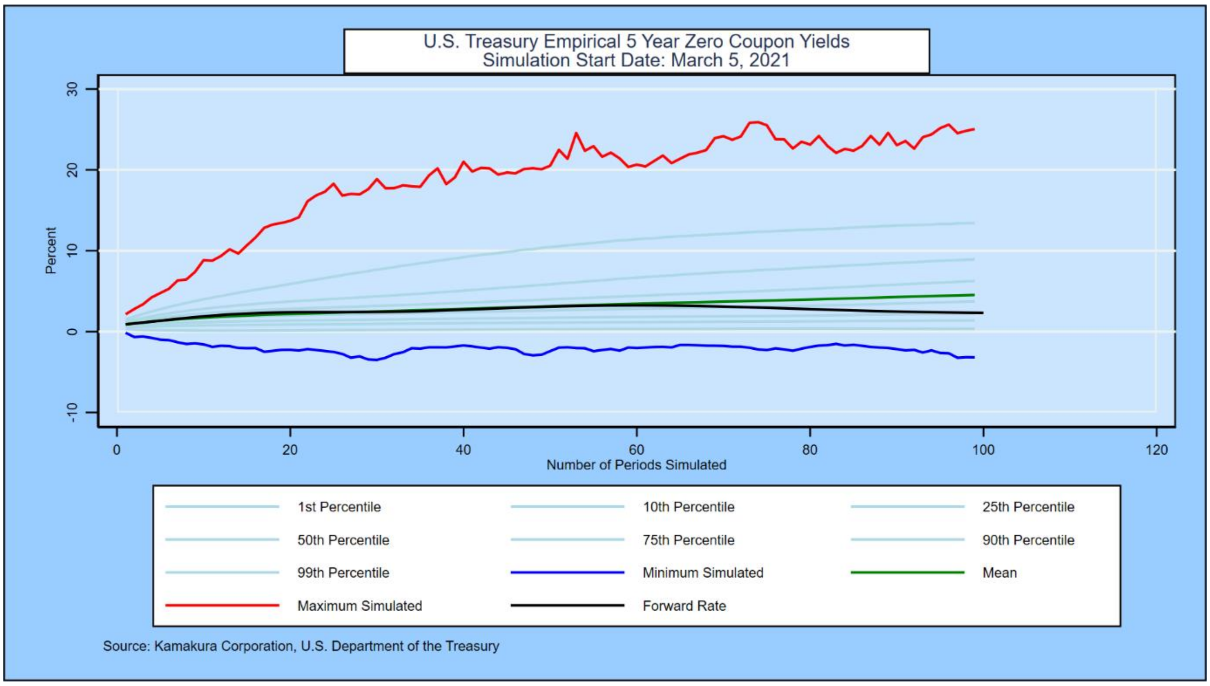


Exhibit XXVII: 10-year Yields

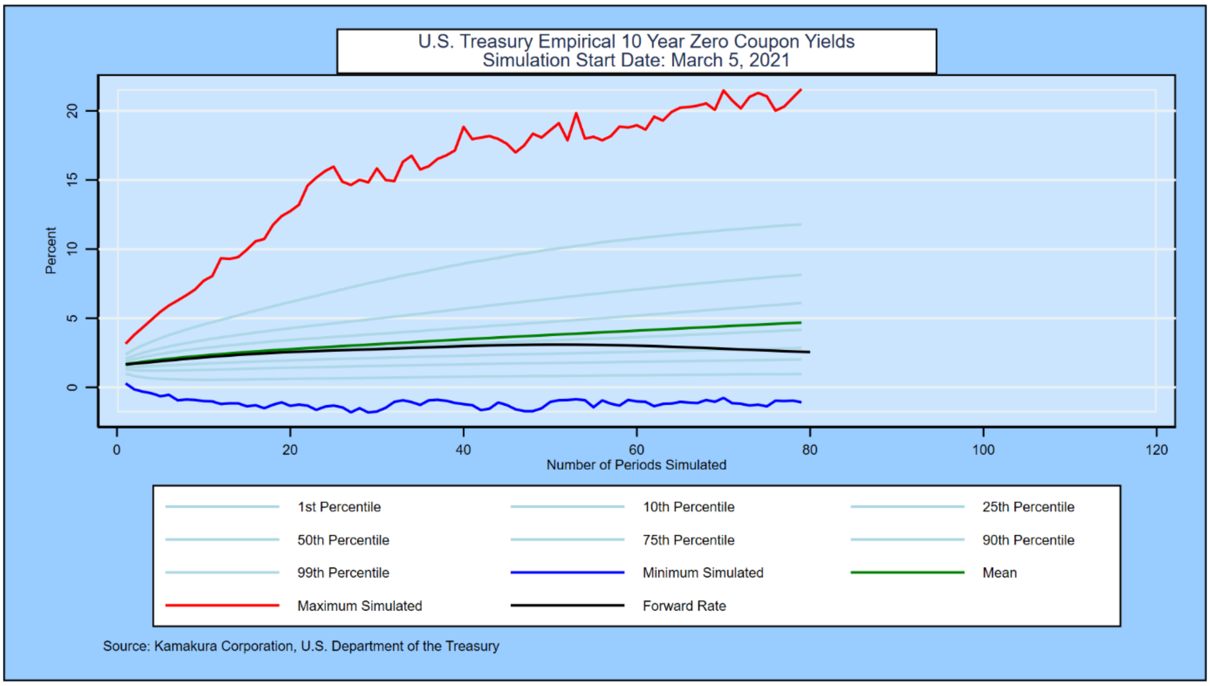
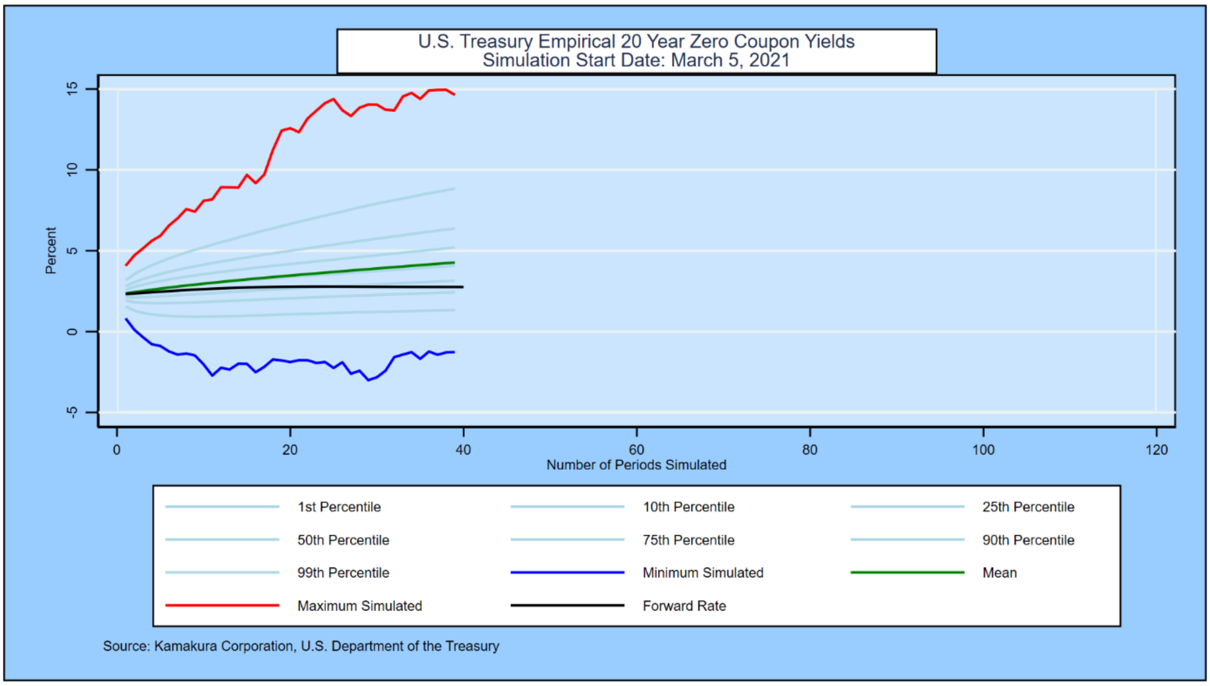


Exhibit XXVIII: 20 Year Yields

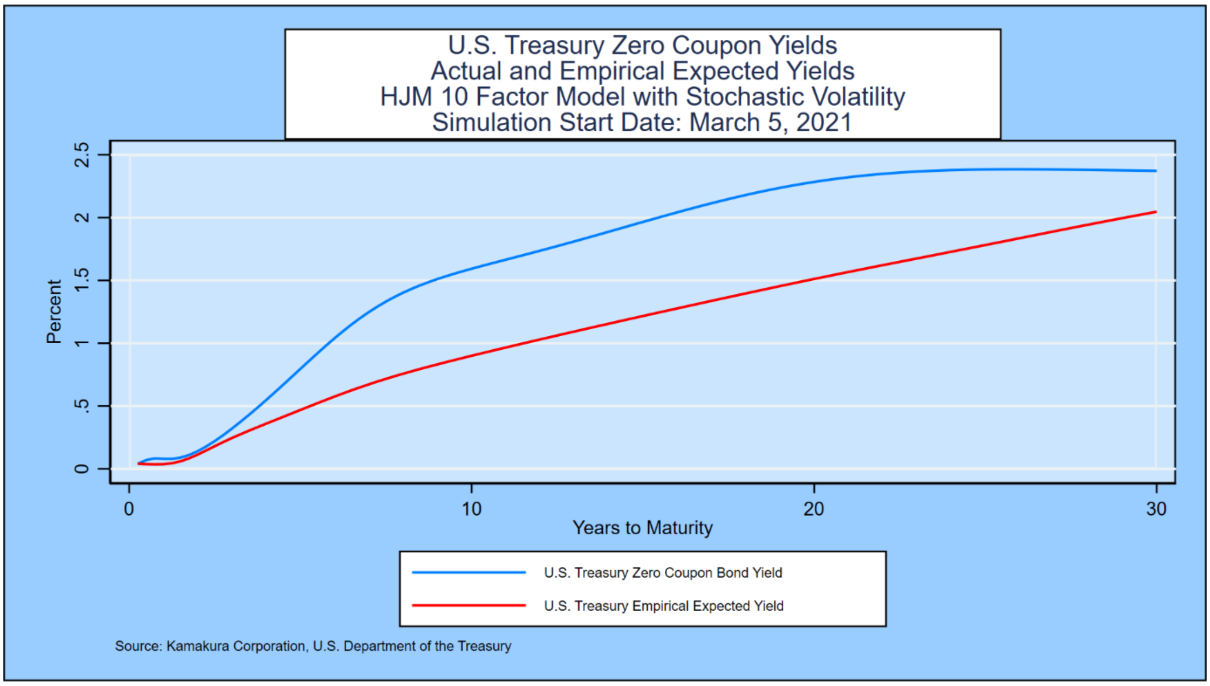


We again determine that there are no unexpected variations in the distribution of empirical yields over time.

D. Simulation of the Term Premium

The size of the "term premium" of actual zero-coupon yields over the expected level of the short rate is a topic of great interest to both academics and policy makers. In a stochastic volatility model, the term premium must be determined by simulation because in general there is no closed form solution for expected future rates. The table below shows a term premium that widens initially, then narrows gradually as the simulation proceeds over time.

Exhibit XXIX: Simulation of the Term Premium

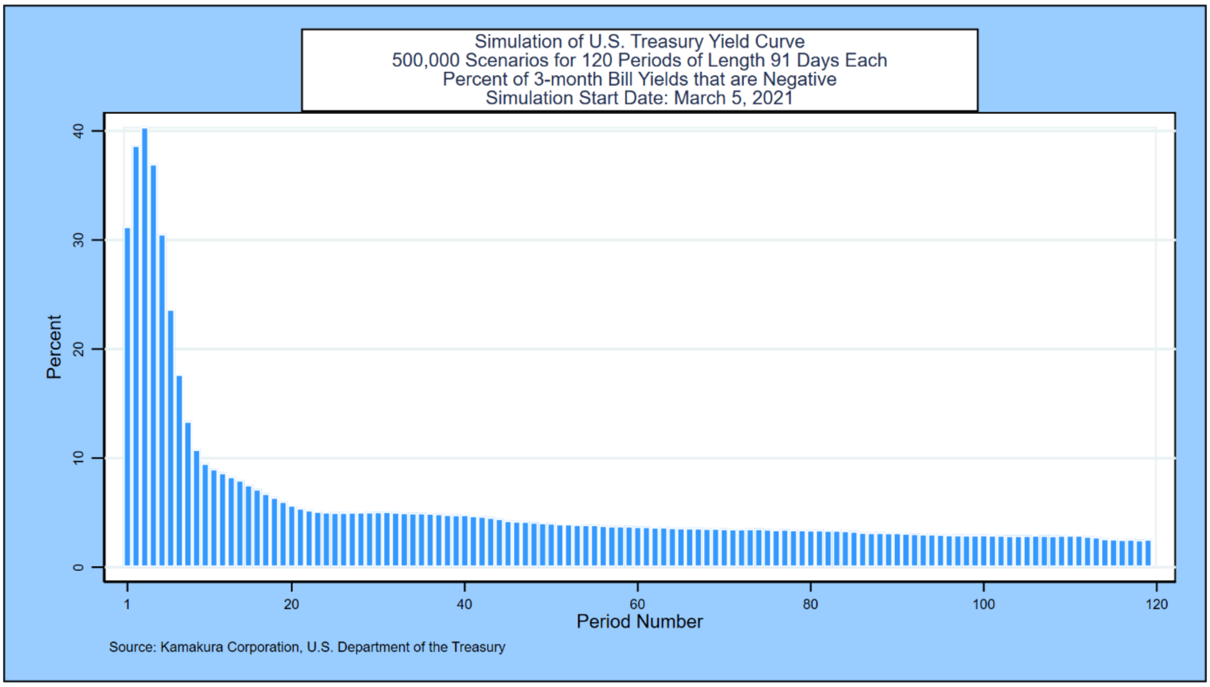


We again conclude that the simulation produces results that are consistent with the "scientific knowledge" about the variation in interest rates around the world.

E. Probability of Negative Short-term Interest Rates

Exhibit XXX shows the simulated probability of negative empirical 3-month Treasury bill rates.

Exhibit XXX

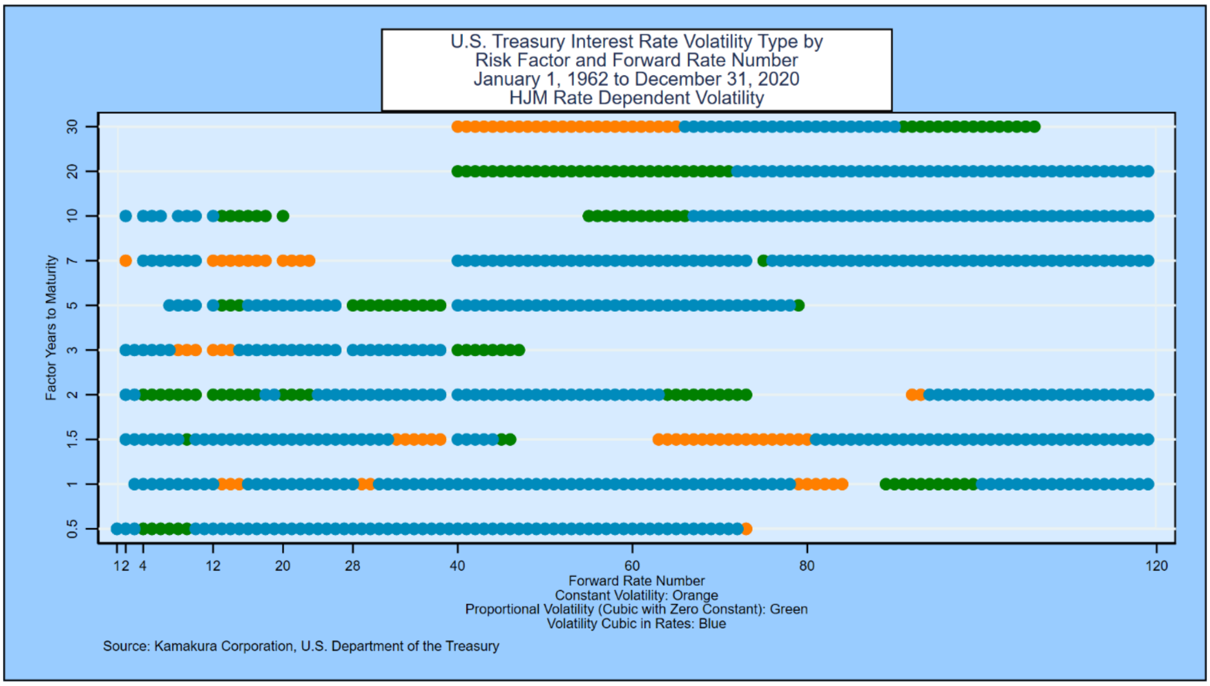


IV. Conclusion

The U.S. Treasury yield curve is driven by 10 factors, a number of factors very similar to government yield curves in 12 other markets for which studies have been conducted. The January 2, 1962 to December 31, 2020 yield history for the United States is both relatively long and spans a wide range of interest rate experience.

The stochastic volatility assumption provides more accurate and more reasonable parameters than a constant volatility model, particularly in the context of Bayesian simulations as part of the model validation process. Exhibit XXXI summarizes the reasons for those conclusions:

Exhibit XXXI: Statistical Significance Summarize and Volatility Classification



The vertical axis lists the maturities used as risk factors by years to maturity of the underlying quarterly forward rate. The risk factors are the idiosyncratic movement of each of these forward rates. If the risk factor is statistically significant in explaining the movement of forward rates with the quarterly maturities listed on the horizontal axis, a dot is placed in the grid. Note that the quarterly forward rates maturing in 20 and 30 years are only used as explanatory variables for maturities of 10 years and longer.

The nature of interest rate volatility for each combination of risk factor maturity and forward rate maturity is color coded. If the derived volatility is constant, the color code is orange. This is the affine specification. The graph shows immediately that a small minority of the risk factor maturity/forward rate maturity volatilities is consistent with the affine structure. The green and blue codes address the issue of whether interest rate volatility for that combination of risk factor maturity and forward rate maturity is zero or not when the forward rate is zero. If the measured volatility at a zero forward rate level is zero, the color code is green. Otherwise, the color code is blue. In both cases, the volatility is a stochastic function of the forward rates at the start of the simulation period.

The chart summarizes the fact that all 10 factors are statistically significant across the yield curve for U.S. Treasuries. The dominant derived interest rate volatility is the cubic stochastic volatility specification with a non-zero constant. An affine assumption for interest rate volatility is best fitting for a small minority of the combinations of risk factor maturity and forward rate maturity.

Appendix

In spite of the overwhelming evidence across countries that government bond yields are driven by multiple factors, the use of single factor term structure models in interest rate risk management systems remains common even in some of the world's largest banks. This appendix asks and answers a number of important questions on the use of one factor models that any sophisticated model audit would pose. Given the answers below, most analysts would conclude that one factor term structure models are less accurate than a long list of multi-factor term structure models and that the one factor models would therefore fail a model audit.

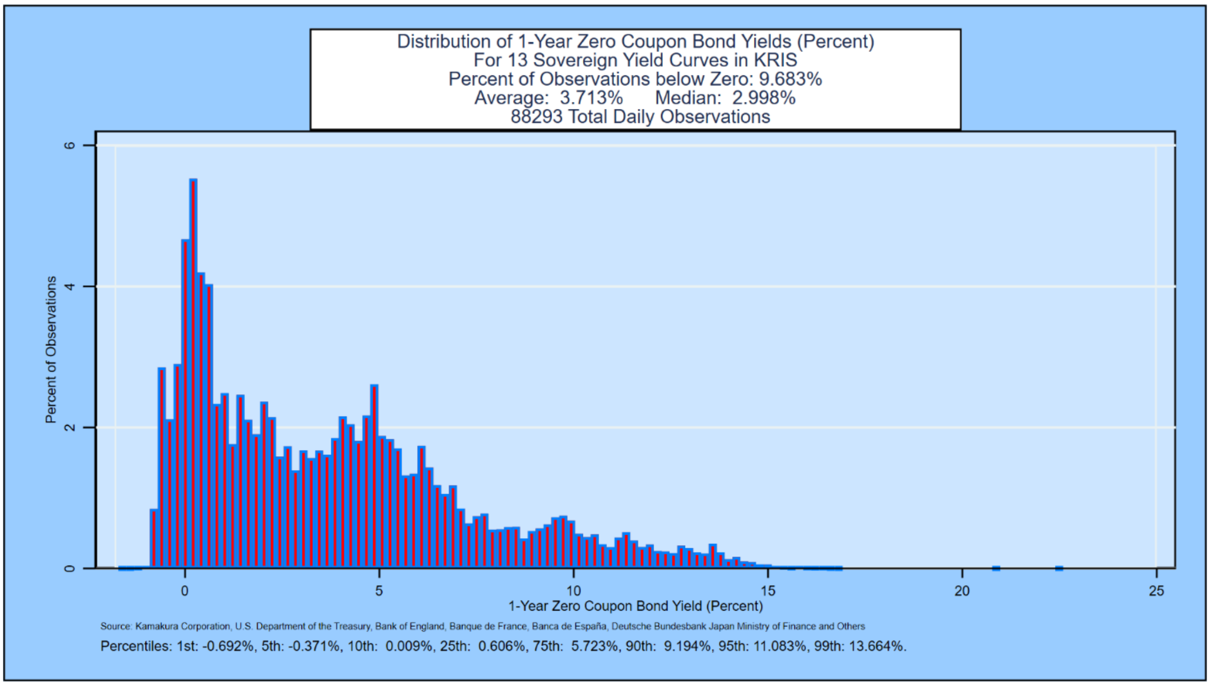
We address two classes of one factor term structure models, all of which are special cases of the Heath, Jarrow and Morton framework, in this appendix using data from the U.S. Treasury market. Answers for other government bond markets cited in the references are nearly identical.

One factor models with rate-dependent interest rate volatility;
- Cox, Ingersoll and Ross (1985)
- Black, Derman and Toy (1990)
- Black and Karasinski (1991)

One factor models with constant interest rate volatility (affine models)
- Vasicek (1977)
- Ho and Lee (1986)
- Extended Vasicek or Hull and White Model (1990, 1993)

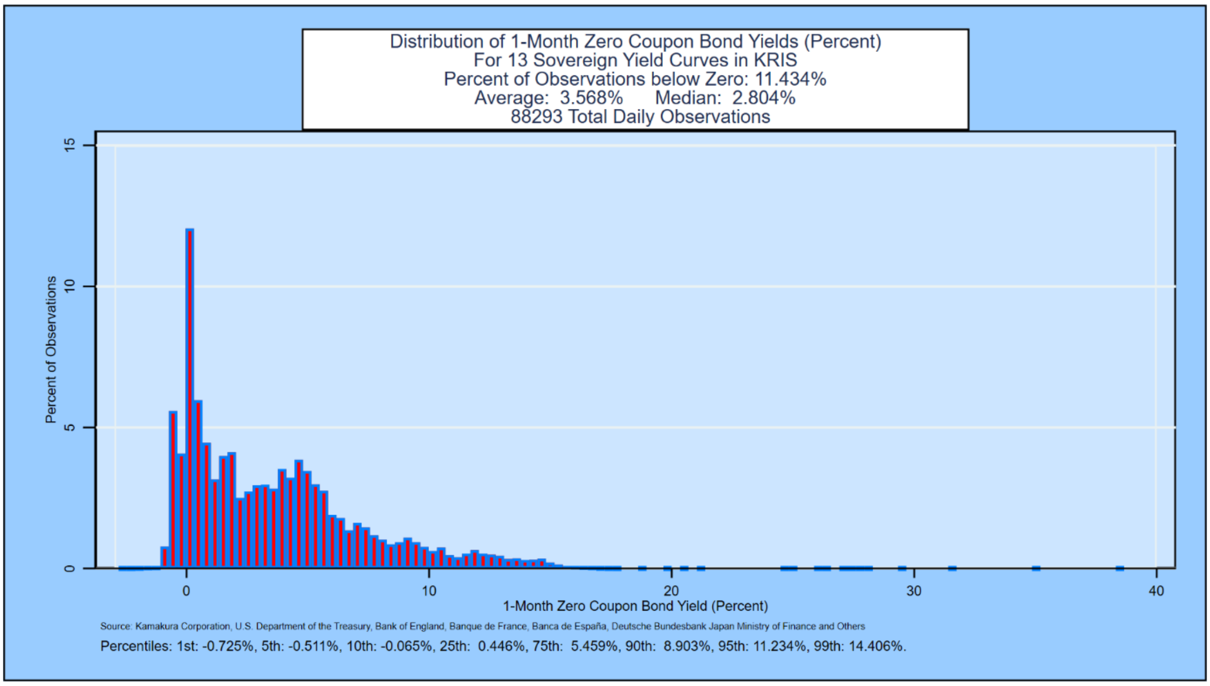
Non-parametric test 1: Can interest rates be negative in the model?

The one factor models with rate-dependent interest rate volatility make it impossible for interest rates to be negative. Is this implication true or false? It is false, as this histogram of 1-year zero-coupon bond yields from government securities in 13 countries confirms. 9.683% of observations were negative:



Non-parametric test 2: As commonly implemented, one factor term structure models imply that all yields will either (a) rise, (b) fall, or (c) remain unchanged. This implication is false, as documented for the United States in Table II. In fact, yield curves have twisted on 83.65% of the observations for the U.S. Treasury market.

Non-parametric test 3: The constant coefficient one-factor models imply that zero-coupon yields are normally distributed and so are the changes in zero-coupon yields. In the U.S. Treasury market, this implication is rejected by three common statistical tests for 120 of 120 quarterly maturities for zero yields and for all 120 of the quarterly changes. This histogram of 1-month zero-coupon bond yields from government bond markets in 13 countries provides visual confirmation that normality is a poor approximation to the probability distribution of interest rates.



Assertion A: There are no factors other than the short-term rate of interest that are statistically significant in explaining yield curve movements. This assertion is false. Table VI shows, using principal components analysis, that 8-12 factors are needed to explain the movements of the U.S. Treasury yield curve. Exhibit XXXI makes the same point in more detail.

Assertion B: There may be more than one factor, but the incremental explanatory power of the 2nd and other factors is so miniscule as to be useless. This assertion is false, as the 2nd through 10th factors in the U.S. Treasury market explain 47% of forward rate movements, compared to 53.5% for the first factor alone. In most countries, the best "first factor" is not the short rate of interest used by many large banks; it is the parallel shift factor of the Ho and Lee model.

Assertion C: A one-factor "regime shift" model is all that is necessary to match the explanatory power of the 2nd and other factors. This assertion is also false. A recent study prepared for a major U.S. bank regulator also confirmed that a one factor "regime shift" term structure model made essentially no incremental contribution toward resolving the persistent lack of accuracy in one factor term structure models.

Finally, a Kamakura Corporation study comparing a 1-factor and 10-factor Heath Jarrow and Morton simulation results shows that the simulated volatility of interest rates is underestimated by 61% to 83%. Moreover, the average level of simulated

U.S. Treasury yields is biased lower. Finally, the probability of negative rates implied by the 1-factor model is also biased low as shown here:

Kamakura Corporation
Comparison of Simulations for Different Term Structure Models
Base Model: U.S. Treasury Heath Jarrow and Morton 10-Factor Model, 1962-2020
Comparison Model: U.S. Treasury Heath Jarrow and Morton 1-Factor Model, 1962-2020
Starting Yield Curve: U.S. Treasury yields, March 5, 2021
Scenarios: 500,000
Time Horizon: 30 years
All Yields are Observable (Empirical) Yields in Percent

		Zero Coupon Bond Yields					
Yield Maturity	Simulated Years Forward	Base Mean Yield	Comparison Mean Yield	Absolute Difference	Base Standard Deviation	Compare Standard Deviation	Percent Difference
3 months	1	0.076	0.083	0.007	0.232	0.090	-61.2%
3 months	5	1.041	0.887	-0.154	1.003	0.247	-75.4%
3 months	10	1.588	1.114	-0.474	1.866	0.426	-77.2%
3 months	15	2.151	1.283	-0.868	2.580	0.580	-77.5%
5 years	1	1.244	1.238	-0.006	0.494	0.082	-83.4%
5 years	5	2.193	1.990	-0.203	1.190	0.254	-78.7%
5 years	10	2.815	2.299	-0.516	1.790	0.378	-78.9%
5 years	15	3.446	2.480	-0.966	2.349	0.452	-80.8%
10 years	1	1.922	1.913	-0.009	0.570	0.131	-77.0%
10 years	5	2.760	2.561	-0.199	1.160	0.271	-76.6%
10 years	10	3.468	2.897	-0.571	1.691	0.309	-81.7%
10 years	15	4.100	3.080	-1.020	2.136	0.343	-83.9%
		Par Coupon Bond Yields					
3 months	1	0.076	0.083	0.007	0.232	0.090	-61.2%
3 months	5	1.044	0.888	-0.156	1.009	0.247	-75.5%
3 months	10	1.596	1.116	-0.480	1.887	0.427	-77.4%
3 months	15	2.165	1.286	-0.879	2.615	0.584	-77.7%
5 years	1	1.235	1.230	-0.005	0.488	0.082	-83.2%
5 years	5	2.189	1.984	-0.205	1.196	0.255	-78.7%
5 years	10	2.816	2.292	-0.524	1.821	0.382	-79.0%
5 years	15	3.459	2.475	-0.984	2.408	0.461	-80.9%
10 years	1	1.874	1.867	-0.007	0.551	0.124	-77.5%
10 years	5	2.711	2.513	-0.198	1.147	0.266	-76.8%
10 years	10	3.405	2.844	-0.561	1.695	0.316	-81.4%
10 years	15	4.401	3.024	-1.377	2.181	0.359	-83.5%

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