The Valuation of Corporate Coupon Bonds*

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Abstract

This paper proposes and estimates a tractable, arbitrage-free valuation model for corporate coupon bonds that includes a more realistic recovery rate process. The existing empirical literature uses a recovery rate process that is misspecified because it includes recovery rates for coupons due after default. Misspecification errors resulting from assuming recovery on all coupons can be substantial in size. They are larger if recovery rates, coupons, maturity and default probabilities are larger. We present evidence that coupon bond market transaction prices reflect the different recovery rates that our model predicts and that our model provides a good fit to market prices.

1 Introduction

An important and still debated issue in the fixed income literature is the exact decomposition of a coupon bond's credit spread into its various components: the expected loss, a default risk premium, a liquidity risk premium, and an adjustment for the deductibility of government bond income for state taxes. This literature can be partitioned into two streams. The first stream estimates credit spreads directly (see Elton, Gruber, Agrawal, and Mann [18], Collin-Dufresne, Goldstein, and Martin [13]), and the second stream prices bonds or related securities using a reduced form model (see Duffee [15], Duffie, Pedersen, and Singleton [16], Driessen [14], and Longstaff, Mithal, and Neis [23]). A careful reading of these papers shows

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that this literature makes the assumption (either implicitly or explicitly) that a coupon bond is equivalent to a portfolio of risky zero-coupon bonds valued using a single term structure. The number of zero-coupon bonds held in the portfolio corresponds to the promised coupons and principal with their maturities corresponding to the payment dates (see expression (10) in the text). For the credit spread estimation literature, this implicit assumption follows because all promised coupons and principal are included when computing a bond's credit spread. In the reduced form model literature, the recovery rate process utilized is the "recovery of market value (RMV)" introduced by Lando [33] and Duffie and Singleton [17], which implies this result. This pricing approach assumes that when discounting, coupon and principal cash flows are treated the same, and, therefore, that both promised payments entitle the holder to recovery in default. For subsequent discussion, we therefore call this approach the "full-coupon recovery" model.

As shown by Jarrow [25], a single term structure of risky zero-coupon bonds that can be used for valuing coupon bonds is valid if and only if all of the risky zero-coupon bonds are of equal seniority and all have the same recovery rate in the event of default. However, this assumption is inconsistent with industry practice. After default, as evidenced by financial restructurings and default proceedings, only the bond's principal becomes due, and no additional coupon payments are made on or after the default date. This implies that coupon and principal payments cannot be valued using the same (single) credit spread or spread term structure and that basing a bond valuation model on the erroneous assumption of equal seniority will produce predicted model prices that have misspecification errors.

Industry practice has been confirmed in the recovery rate estimation literature where it has been shown that alternative recovery rate processes,[1] either the "recovery of face value (RFV)" or the "recovery of Treasuries (RTV)" formulations, provide a better approximation to realized recovery rates than does RMV (see Guha and Sbuelz [21] and Guo, Jarrow and Lin [20]). And, it is also well known that both the RFV and RTV recovery rate processes are consistent with a zero recovery on coupons promised after default. Therefore, these recovery rate processes do not imply the full-coupon recovery model. See Jarrow and Turnbull [32], Bielecki and Rutkowski [5], chapter 13, and Collin-Dufresne and Goldstein [12] for models with zero recovery on coupons promised after default.[2]

The purpose of this paper is to explore, both theoretically and empirically, the effect on bond prices assuming zero recovery on coupons promised after default. To do so we derive an empirically tractable reduced form bond pricing model, the form of which is new to the literature. For subsequent discussion, we refer to it as the "reduced form" model. We derive an intuitive and straightforward-

[1]See Bielecki and Rutkowski [5], Chapter 8 for a discussion of these different recovery rate processes.

[2]Unlike our paper, these studies do not explore empirically the pricing effect of zero coupon recovery.

to-implement pricing formula in which model prices depend only on the risk-free term structure, the term structure of default probabilities, and two parameters that we estimate – the recovery rate and a parameter capturing liquidity. We also show that the bond can be valued using the following "building block" securities: zero recovery zero coupon bonds, which pay off if there is no default, and digital recovery bonds, which pay off only in the event of default. The decomposition is useful both to build pricing intuition and for the empirical implementation.

We adapt the common practice of pricing bonds by calculating credit spreads and show that our reduced form model can be computed using two different issuer and maturity specific discount rate functions – spread curves – one for coupons and one for principal, rather than using the traditional single curve for both.

We perform a calibration of the model to demonstrate that the spreads appropriate for discounting coupon and principal payments can be quite different. We calculate misspecification errors relative to a full-coupon recovery model, which generates prices that are too large since it erroneously assumes a positive recovery associated with coupon payments due after default, when in reality they are zero. The misspecification errors that result from this assumption are larger if recovery rates, default probabilities, maturity or coupon payments are larger. For example, for a 10-year bond with a recovery rate of 40%, a coupon of 2.61%, and an annual default probability of 1%, the full-coupon recovery model will assign a price that is \$0.50 too large. If it is a 30-year bond, the price is \$3.61 too large, a substantial difference relative to the correct price, which is equal to par. We calculate exact misspecification errors and also provide an approximate formula that can be used to estimate the misspecification error magnitudes. In this estimate, misspecification errors are proportional to the recovery rate and the coupon size; they are approximately proportional to the default probability and the square of the number of coupon payments, which results in a close relationship to maturity.

We begin our empirical work by providing direct evidence of a difference in seniority between principal and coupons. We provide three examples of issuers that have filed for bankruptcy: Lehman Brothers, Pacific Gas and Electric (PG&E), and Weatherford International. We use both the misspecified full-coupon recovery model and our reduced form model to price the bonds. We find that pricing errors from using the full coupon recovery model are between five and ten times as large as the reduced form model pricing errors. Observed prices are thus consistent with market participants assuming zero recovery on coupons and they are inconsistent with the assumption of equal recovery. This analysis provides independent evidence in support of the validity of the industry pricing practice implying zero recovery on coupons discussed above.

Next, we investigate if and when the differences in seniority is reflected in traded bond prices prior to default. To do so we perform a comparative analysis of the reduced form model against the full-coupon recovery model and a credit ratings-based valuation model that underlies the Basel Committee on Banking Supervision's regulations (Basel Committee on Banking Supervision [2, 3]). Both of these models value a coupon bond as if it is a portfolio of zero-coupon bonds (as discussed above). Our sample consists of daily market prices for a collection of liquidly traded bonds over the period from September 1, 2017 through June 30, 2019.

We show that the reduced form model outperforms both the full coupon recovery and the ratings based models. First, we fit the reduced form model to the data to recover unbiased estimates of the model parameters. Second, we calculate predicted full-coupon recovery model misspecification errors based on these parameters. Misspecification errors are large (5% are larger than \$1.43) and they are highly correlated with our simple approximation formula. We next run a horse race between the models; we use both models for pricing and then compare pricing errors. The reduced form model again outperforms the coupon recovery model. This is true for the full sample. In particular, the outperformance is larger for large default probability issuer-days, exactly those cases where we expect the erroneous assumption of equal seniority to have the largest impact on misspecification errors. The outperformance is also larger on those days where the full-coupon recovery model's misspecified assumptions imply that fitting the data becomes more difficult relative to the reduced form model. When including the more restrictive assumption that credit spreads are the same for bonds with the same rating (the credit ratings-based model), performance drops further. In sum, this evidence provides strong support for the necessity of using the alternative methodology of our reduced form model.

The outline of the paper is as follows. Section 2 presents the model for valuing risky coupon bonds, while Section 3 discusses the model's empirical parameterization. Section 4 discusses the estimation procedures, while Section 5 presents some illustrative pricing results for three companies that filed for bankruptcy. Section 6 presents a comparative analysis of two alternative pricing models, Section 7 provides a time series comparison of these models and the ratings-based model, and Section 8 presents some specification tests. Section 9 concludes.

2 The Model

This section presents the model's set-up, which is based on the reduced form model of Jarrow and Turnbull [30], [32][3]. We consider a continuous trading model on a finite horizon $[0, T]$. The uncertainty in the model is characterized by a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$

[3]We note that Jarrow and Turnbull [32] contains a reduced form model with zero recoveries paid on coupons due after default.

satisfies the usual hypotheses and $\mathcal{F} = \mathcal{F}_T$.[4] Here $\mathbb{P}$ is the statistical probability measure. By the statistical probability measure we mean that probability $\mathbb{P}$ from which historical time series data are generated.

2.1 The Default-free Bond Market

We assume that traded in the economy are default-free zero-coupon bonds of all maturities, a default-free money market account, and a risky coupon bond (to be described later). The market is assumed to be frictionless and competitive. By frictionless we mean that there are no transaction costs or trading constraints. By competitive we mean that all traders act as price takers, i.e. their trades have no impact on the price. Both the frictionless and competitive market assumptions are relaxed, subsequently, when we add a liquidity discount to the valuation formula (see expression (11) below). A liquidity discount modifies the valuation formula incorporating transaction costs, convenience yields caused by trading constraints, and any quantity impact of a trade on the market price.

The default-free money market account earns interest continuously at the default-free spot rate of interest, r_t, which is adapted to $(\mathcal{F}_t)_{t\in[0,T]}$. We initialize the money market account with a dollar at time 0 and denote its time t value by[5]

$$B_t = e^{\int_0^t r_s ds}. \tag{1}$$

We let the time t value of a default-free zero-coupon bond paying a dollar at time T be strictly positive and denoted by $p(t,T) > 0$.

2.2 The Risky Coupon Bond

We consider a firm that issues a bond with a coupon of C dollars, a face value equal to L dollars, and a maturity date T. The bond pays the C dollar coupons at intermediate dates $\{t_1, ..., t_m = T\}$, but only up to the default time τ. For notational convenience, let the current time $t = t_0$. If default happens prior to the maturity date in the time interval $(t_{k-1}, t_k]$, then we assume that the bond pays a stochastic recovery rate of $\delta_{t_k} \in [0,1]$ on only the notional of L dollars at time t_k, which is $\mathcal{F}_{t_k}-$measurable. It is important to note that default can happen anytime within this interval, but the payment only occurs at the end. If default does not happen, the face value of L dollars is repaid at time T.

As an approximation, we assume that if default happens within the time interval $(t_{k-1}, t_k]$, then no recovery is received for the coupon payment promised at time t_k. In practice, a portion of the next coupon payment after default represents some accrued interest in an accounting sense, earned, but not yet paid. This

[4] See Protter [35] for the definitions of these various terms.

[5] Of course, we assume the necessary measurability and integrability such that the following expression is well-defined.

accrued interest has a recovery rate associated with it.[6] We ignore this residual payment in the event of default.

We now characterize the firm's default time. Let $\Gamma_t = (\Gamma_1(t), \ldots, \Gamma_n(t))' \in \mathbb{R}^n$ be a collection of stochastic processes characterizing the state of the firm and the market at time t with $\mathcal{F}_t^\Gamma$ representing the filtration generated by the state variables Γ_t up to and including time $t \geq 0$. We assume that these state variables are adapted to $\mathcal{F}_t$, which implies that $\mathcal{F}_t^\Gamma \subset \mathcal{F}_t$. We include the default-free spot rate of interest r_t in this set of state variables, which implies that r_t is also $\mathcal{F}_t^\Gamma$ -measurable. Examples of additional state variables that could be included in this set are measures of economic growth, unemployment rates, and inflation rates.

Let $\lambda : [0,T] \times \mathbb{R}^n \longrightarrow [0,\infty)$, denoted $\lambda_t = \lambda_t(\Gamma_t) \geq 0$, be jointly Borel measurable with $\int_0^T \lambda_t(\mathit{\Gamma}_t)dt < \infty$ a.s. $\mathbb{P}$, and let $N_t \in \{0,1,2,\cdots\}$ with $N_0 = 0$ be a Cox process conditioned on $\mathcal{F}_T^\Gamma$ with $\lambda_t(\mathit{\Gamma}_t)$ its intensity process (see Lando [33]). Finally, let $\tau \in [0,T]$ be the stopping time adapted to the filtration $\mathcal{F}_t$ defined by

$$\tau \equiv inf\left\{t > 0 : \; N_t = 1\right\}.$$

We let τ represent the firm's default time. The function $\lambda_t(\mathit{\Gamma}_t)$ is the firm's default intensity, which can be interpreted as the probability of default over the small time interval $[t, t+\Delta]$ conditional upon no default prior to time t. Intuitively, a Cox process is a point process which, conditional upon the information set generated by the state variables process Γ_t over the entire trading horizon $[0,T]$ behaves like a standard Poisson process. In particular,

$$\mathbb{P}(\tau > \mathcal{T} \left| \mathcal{F}_T^\Gamma \vee \mathcal{F}_t \right.) = e^{-\int_t^{\mathcal{T}} \lambda_u du}$$

and

$$\mathbb{P}(\tau > \mathcal{T} \left| \mathcal{F}_t \right.) = E^{\mathbb{P}}\left[e^{-\int_t^{\mathcal{T}} \lambda_u du} \left| \mathcal{F}_t \right. \right].$$

2.3 No Arbitrage

We want to value the risky coupon bond in an arbitrage-free market. Hence, we add the following assumption. Denote the time $t \leq t_1$ value of the coupon bond as v_t.

Assumption. (Existence of an Equivalent Martingale Measure)

There exists an equivalent probability measure $\mathbb{Q}$ such that

$$\frac{p(t,\mathcal{T})}{B_t} \; for \; all \; \mathcal{T} \in [0,T] \quad and \quad \frac{v_t}{B_t}$$

are $\mathbb{Q}$ martingales.

[6]We appreciate the comments from Morrison & Foerster in this regard.

Equivalent means that both the probability measures $\mathbb{P}$ and $\mathbb{Q}$ agree on zero probability events. It is well known that this assumption implies that the market is arbitrage-free. See Jarrow and Protter [24].

Given an equivalent martingale measure $\mathbb{Q}$, define $\widetilde{\lambda}_t \equiv \lambda_t \kappa_t$ to be the intensity process of the Cox process under $\mathbb{Q}$ where $\kappa_t(\omega) \geq 0$ is a predictable process with $\int_0^T \lambda_t(\Gamma_t)\kappa_t dt < \infty$ a.s. $\mathbb{P}$ (see Bremaud [6], p. 167). The process $\kappa_t(\omega)$ represents a default jump risk premium.

2.4 Risk Neutral Valuation

To value the risky coupon bond, we need one additional assumption. We assume that the market is complete or that enough derivatives trade so that the enlarged market including the traded derivatives is complete (see Jacod and Protter [24] for a set of sufficient conditions on an incomplete market such that the expanded market is complete). Given the trading of credit default swaps on traded bonds, this is a reasonable approximation. By the second fundamental theorem of asset pricing (see Jarrow and Protter [29]), completeness implies that the risk-neutral probability $\mathbb{Q}$ is unique and risk-neutral valuation applies.

Given this structure, we can apply risk-neutral valuation to value the coupon bond's cash flows as in the following expression.

$$\begin{aligned} v_t = \textstyle\sum_{k=1}^m CE^{\mathbb{Q}}\left[1_{\{\tau>t_k\}}e^{-\int_t^{t_k} r_u du}\,|\mathcal{F}_t\right] + LE^{\mathbb{Q}}\left[1_{\{\tau>T\}}e^{-\int_t^{T} r_u du}\,|\mathcal{F}_t\right] \\ + \textstyle\sum_{k=1}^m LE^{\mathbb{Q}}\left[1_{\{t_{k-1}<\tau\leq t_k\leq T\}}\delta_{t_k}e^{-\int_t^{t_k} r_u du}\,|\mathcal{F}_t\right] \end{aligned} \tag{2}$$

This valuation formula reflects the expected discounted value of the random cash flows to the risky coupon bond using the risk-neutral probabilities $\mathbb{Q}$. The cash flows correspond to the coupon payments $C1_{\{\tau>t_k\}}$ for $k = 1, \ldots, m$ and the principal $L1_{\{\tau>T\}}$, but only if no default occurs prior to the payment date, plus the recovery payment on the principal in the event of default within the time interval $(t_{k-1}, t_k]$, $1_{\{t_{k-1}<\tau\leq t_k\leq T\}}\delta_{t_k}L$ summed across all the intervals within the bond's life. The discount rate is the default-free spot rate r_t. The adjustment for risk is via the use of the risk-neutral probabilities, instead of the statistical probabilities $\mathbb{P}$. The risk-neutral probabilities include the required risk premium. Note as discussed above that no recovery payments are included for any coupon payments occurring after the default date.

3 The Empirical Parameterization

To empirically implement and estimate expression (2), we need to add more structure to the evolution of the default-free term structure of interest rates, the default process, and the recovery rate process. This section adds this structure.

3.1 The Default and Recovery Rate Processes

This section presents the additional structure imposed on the default and recovery rate process for empirical estimation. For this structure we need the following notation. Let $\mathcal{F}_t^r$ denote the filtration generated by r_t, $\mathcal{F}_t^\delta$ the filtration generated by δ_t, and $\mathcal{F}_\tau$ the $\sigma-$algebra generated by the default time τ (see Protter [35], p. 5). Finally, let $\mathcal{G}_t$ be a filtration independent of $\mathcal{F}_t^r \vee \mathcal{F}_t^\delta \vee \mathcal{F}_\tau$ where $\mathcal{F}_t \equiv \mathcal{F}_t^r \vee \mathcal{F}_t^\delta \vee \mathcal{F}_\tau \vee \mathcal{G}_t$. We can now state our first assumption on these processes.

Assumption. (The Default, Recovery, and Interest Rate Processes)
The default-free spot rate r_t, the default time τ, and the recovery rate process δ_t are independent under $\mathbb{Q}$ given the filtration $\mathcal{G}_t$ for all t.

This assumption is imposed for analytic tractability.[7] It states that under the risk neutral probabilities $\mathbb{Q}$ ($\mathbb{Q}$ provides an adjustment for risk), the spot rate, the default time, and the recovery rate processes are independent given the information in the filtration $\mathcal{G}_t$, where $\mathcal{G}_t$ is the information in $\mathcal{F}_t$ that is independent of the information that these processes themselves generate. The information in $\mathcal{G}_t$ includes, for example, macro-economic variables characterizing the state of the economy and a firm's balance sheet data. Holding the information in $\mathcal{G}_t$ constant, this implies that these processes exhibit no conditional (or unconditional) correlations across time under the risk neutral probabilities. This is a weak assumption on the evolutions of the default-free spot rate, the default time, and the recovery rate.

If fact, this assumption imposes very little structure on the evolutions of these stochastic processes under the statistical probabilities $\mathbb{P}$. Under the statistical probabilities, these processes need not be independent given $\mathcal{G}_t$. Hence, nonzero pairwise correlations between the observed default-free spot rate, the default time, and the recovery rate processes, which are realizations under the statistical probabilities $\mathbb{P}$, are not excluded by this assumption. Non-zero correlations across the default-free spot rate, default times, and recovery rates have been observed in historical data, and these observations are consistent with the previous assumption.

Next, we add the following assumption on the recovery rate process.

Assumption. (The Recovery Rate Process is a Martingale under $\mathbb{Q}$)

$$E^{\mathbb{Q}}\left[\delta_s \left| \mathcal{F}_t \right.\right] = \delta_t$$

This assumption states that, after adjusting for risk, the best estimate at time t of the recovery rate process at time s is its value at time t. This implies that

[7] In the subsequent proofs, this will enable the expectation of a product, given $\mathcal{G}_t$, to be the product of the expectations.

the recovery rate process is stochastic and changes across time given changes in the information set $\mathcal{F}_t$. For example, recovery rates can be lower in bad times than in good times. We note this assumption does not imply that the recovery rate process is a martingale under the statistical probabilities $\mathbb{P}$. It is important to note that the recovery rate process δ_t represents the time t "expected value" of the recovery rate δ_{t_k} for default within $(t_{k-1}, t_k]$, after the adjustment for risk. The risk adjustment (under $\mathbb{Q}$) will typically lower the estimated value of the recovery rate. Hence, δ_t does not represent the actual recovery rate at time t. This observation will prove useful in a subsequent section.

3.2 Zero-Recovery and Digital Recovery Securities

Before simplifying expression (2), we first digress to talk about two more basic credit risky securities, zero-recovery zero-coupon bonds and digital recovery bonds. These more basic securities will enable us to provide an intuitive interpretation of the subsequent valuation formula. These securities were first discussed in the literature by Madan and Udal [34].

Consider the time interval $[t_{k-1}, t_k]$ for $k = 0, \ldots, m$,[8] where default has not yet happened, i.e. $\tau > t_{k-1}$. Define

- (A Zero-recovery Zero-coupon Bond with maturity t_k) A zero-recovery zero-coupon bond with maturity t_k pays \$1 at time t_k only if default occurs after t_k, i.e. in symbols its payoff is $1_{\{\tau > t_k\}}$. The value of this security at time $s \geq t_{k-1}$ is
$$E^{\mathbb{Q}}\left[1_{\{\tau > t_k\}} e^{-\int_s^{t_k} r_u du} | \mathcal{F}_s\right]. \tag{3}$$
Hence, these are risky zero-coupon bonds that have a zero recovery rate in the event of default.

- (A Digital Recovery Bond for the interval $(t_{k-1}, t_k]$) A digital recovery bond pays \$1 at time t_k if default occurs within $(t_{k-1}, t_k]$. The value of this security at time $s \geq t_{k-1}$ is
$$E^{\mathbb{Q}}\left[1_{\{\tau \leq t_k\}} e^{-\int_s^{t_k} r_u du} | \mathcal{F}_s\right]. \tag{4}$$

Using the identity $1 = 1_{\{\tau \leq t_k\}} + 1_{\{\tau > t_k\}}$, it follows that

$$E^{\mathbb{Q}}\left[e^{-\int_s^{t_k} r_u du} | \mathcal{F}_s\right] = E^{\mathbb{Q}}\left[1_{\{\tau \leq t_k\}} e^{-\int_s^{t_k} r_u du} | \mathcal{F}_s\right] + E^{\mathbb{Q}}\left[1_{\{\tau > t_k\}} e^{-\int_s^{t_k} r_u du} | \mathcal{F}_s\right]$$

or

$$p(s, t_k) = E^{\mathbb{Q}}\left[1_{\{\tau \leq t_k\}} e^{-\int_s^{t_k} r_u du} | \mathcal{F}_s\right] + E^{\mathbb{Q}}\left[1_{\{\tau > t_k\}} e^{-\int_s^{t_k} r_u du} | \mathcal{F}_s\right]. \tag{5}$$

[8] Note that when $k = m$, $t_m = T$.

In words, at time $s \in [t_{k-1}, t_k]$ where default has not yet happened ($\tau > t_{k-1}$), a default-free zero coupon bond is equivalent to a portfolio consisting of a digital recovery bond for the interval $(t_{k-1}, t_k]$ and a zero-recovery zero-coupon bond with maturity t_k.[9]

The time t value of the left and right sides of expression (5) is

$$\begin{aligned} E^{\mathbb{Q}}\left[1_{\{\tau>t_{k-1}\}}p(s,t_k)e^{-\int_t^s r_u du}\,|\mathcal{F}_t\right] = \quad & E^{\mathbb{Q}}\left[1_{\{t_{k-1}<\tau\leq t_k\}}e^{-\int_t^{t_k} r_u du}\,|\mathcal{F}_t\right] \\ & +E^{\mathbb{Q}}\left[1_{\{\tau>t_k\}}e^{-\int_t^{t_k} r_u du}\,|\mathcal{F}_t\right]. \end{aligned} \tag{6}$$

This represents the present value of the t_k maturity zero-coupon bond at time s, which is received only if there is no default before or at time t_{k-1}. The right side is the present value of the digital recovery bond for the interval $(t_{k-1}, t_k]$ and the zero-recovery zero-coupon bond with maturity t_k. This expression is used below in the derivation of the empirical valuation formula.

3.3 The Estimated Valuation Formula

This section derives the estimated valuation formula used in the estimation under the above assumptions. From expression (2), using the independence assumption, the recovery rate assumption, and the definition of the risk neutral probability $\mathbb{Q}$, we have that

$$\begin{aligned} v_t = \textstyle\sum_{k=1}^m Cp(t,t_k)E^{\mathbb{Q}}\left[1_{\{\tau>t_k\}}\,|\mathcal{F}_t\right] + Lp(t,T)E^{\mathbb{Q}}\left[1_{\{\tau>T\}}\,|\mathcal{F}_t\right] \\ +\textstyle\sum_{k=1}^m Lp(t,t_k)\delta_t E^{\mathbb{Q}}\left[1_{\{t_{k-1}<\tau\leq t_k\leq T\}}\,|\mathcal{F}_t\right]. \end{aligned} \tag{7}$$

Using the identity $1_{\{t_{k-1}<\tau\leq t_k\}} = 1_{\{\tau>t_{k-1}\}} - 1_{\{\tau>t_k\}}$, after some algebra we get that

$$\begin{aligned} v_t = Lp(t,T)E^{\mathbb{Q}}\left[1_{\{\tau>T\}}\,|\mathcal{F}_t\right] + (C - L\delta_t)\textstyle\sum_{k=1}^m p(t,t_k)E^{\mathbb{Q}}\left[1_{\{\tau>t_k\}}\,|\mathcal{F}_t\right] \\ +L\delta_t \textstyle\sum_{k=1}^m p(t,t_k)E^{\mathbb{Q}}\left[1_{\{\tau>t_{k-1}\}}\,|\mathcal{F}_t\right]. \end{aligned} \tag{8}$$

Using expression (6), we see that the last term in this expression is the sum of the present values of the t_k maturity zero-coupon bonds, which are received only if there is no default before or at time t_{k-1}. This expression has an intuitive interpretation, using the zero-recovery zero-coupon bonds previously discussed. It shows that a risky coupon bond can always be decomposed into a portfolio of zero-recovery zero-coupon bonds.

[9]The reason for this is that, as discussed above, there is a close relationship between the value of digital recovery securities and adjacent zero-recovery zero-coupon bonds. We also note that the valuation of corporate coupon bonds is only one example of the advantages of pricing using these building block securities. Other securities that depend on default, e.g. CDS contracts, sovereign bonds, and "bail in" bonds can also be priced using the building block securities. We leave application of this methodology to the pricing of these securities to future research.

Defining the probability

$$Q(t, t_{k-1}) \equiv \mathbb{Q}(t < \tau \leq t_{k-1} \,|\mathcal{F}_t\,) = 1 - E^{\mathbb{Q}}\left[1_{\{\tau > t_{k-1}\}}\,|\mathcal{F}_t\right],$$

which is the risk neutral probability at time t that default occurs between times t and t_{k-1}, we can rewrite (8) in an another form,

$$\begin{array}{c} v_t = Lp(t,T)\left[1 - Q(t,T)\right] + (C - L\delta_t)\sum_{k=1}^{m} p(t,t_k)\left[1 - Q(t,t_k)\right] \\ +L\delta_t \sum_{k=1}^{m} p(t,t_k)\left[1 - Q(t,t_{k-1})\right]. \end{array} \tag{9}$$

This expression facilitates the computation utilized below. Indeed, the default-free zero-coupon bond prices $p(t, \mathcal{T})$ for $\mathcal{T} \in [t, T]$ are (conceptually) observable and the intensity process embedded in the risk neutral default probabilities

$$Q(t, \mathcal{T}) = E^{\mathbb{Q}}\left[e^{-\int_t^{\mathcal{T}} \lambda_u du}\,|\mathcal{F}_t\right]$$

can be estimated using historical time series data (see the diversifiable default risk assumption below).

In addition, in this form it is easy to see that the value of this coupon bond is not equal to the sum of the coupons and principal times the value of a collection of zero-coupon bonds from a single risky term structure with the appropriate maturity dates and that pay the fractional recovery rate of δ_t in the event of default. Let $D(t, t_k)$ denote the time t value of such a zero-coupon bond promising to pay a dollar at time t_k for $k = 1, \ldots, m$. Then, using the same mathematics as above, it can be shown that

$$\begin{array}{l} \sum_{k=1}^{m} CD(t,t_k) + LD(t,T) = \\ \qquad Lp(t,T)\left[1 - Q(t,T)\right] + (C - L\delta_t)\sum_{k=1}^{m} p(t,t_k)\left[1 - Q(t,t_k)\right] \\ \qquad +L\delta_t \sum_{k=1}^{m} p(t,t_k)\left[1 - Q(t,t_{k-1})\right] \\ \qquad + \sum_{k=1}^{m} C(m+1-k)\delta_t p(t,t_k)[Q(t,t_k) - Q(t,t_{k-1})]. \end{array} \tag{10}$$

This expression is called the "full-coupon recovery model." The difference between this model and the zero coupon recovery model is that expression (10) contains terms omitted in expression (9). The additional terms in the full-coupon recovery model, expression (10), are the recovery values for the coupons that would have been paid after the default date, i. e. $\sum_{k=1}^{m} C(m+1-k)\delta_t p(t,t_k)[Q(t,t_k) - Q(t,t_{k-1})]$.[10] These terms are included because all cash flows are discounted using a single risky term structure of zero-coupon bonds.

Lastly, to facilitate the estimation of the intensity process, we assume that default risk is diversifiable in the sense of Jarrow, Lando, and Yu [27].

[10] This term follows because if default occurs during the time interval $(t_{k-1}, t_k]$, the remaining future coupons are $\sum_{j=k}^{m} C = (m+1-k)C$. In the full coupon recovery model, one gets a recovery payment on all the remaining coupons.

Assumption. (Diversifiable Default Risk)

$$\tilde{\lambda}_t(\Gamma_t) = \lambda_t(\Gamma_t)$$

where $\tilde{\lambda}_t(\Gamma_t)$ is the intensity for the Cox process under the risk-neutral probability $\mathbb{Q}$.

This assumption enables the estimation of the default intensities using historical time series data, without the need to adjust the intensity process for a default jump risk premium. It is important to note, as discussed in Jarrow, Lando, and Yu [27], that this assumption does not imply that risky coupon bonds earn no risk premium. Quite the contrary. If the state variables Γ_t driving the default process represent systematic risk, which is the most likely case, then risky coupon bond prices necessarily earn a risk premium due to the bond price's correlation to Γ_t. The diversifiable risk assumption just states that the timing of the default event itself, after conditioning on Γ_t, is diversifiable in a large portfolio. Alternatively stated, in a poor economy all firms are more likely to default. But, the timing of which firms actually default depends on the idiosyncratic risks of the firm's management and operations.

To estimate the default probabilities, we use a proportional hazard rate model (see Fleming and Harrington [19], p. 126), i.e. we assume that

$$\lambda_t(\Gamma_t) = \theta e^{\phi \Gamma_t}$$

where θ is a constant and where ϕ is a vector of constants. Recall that $\Gamma_t = (\Gamma_1(t), \ldots, \Gamma_m(t))' \in \mathbb{R}^m$ are a collection of stochastic processes characterizing the state of the firm and the market at time t. For an application of such a hazard rate model applied to corporate default probabilities see Chava and Jarrow [10].

3.4 A Liquidity Discount

Corporate bond markets are illiquid relative to Treasury bonds or exchange traded equities. This illiquidity implies that corporate bond prices may reflect a liquidity discount (see Jarrow and Turnbull [31], Duffie and Singleton [17], Cherian, Jacquier, and Jarrow [11]). To incorporate such a liquidity discount, we use the discount function $e^{\alpha_t(T-t)}$ to discount the zero-recovery zero-coupon bond component securities that compose a risky coupon bond. We do not apply a liquidity discount to the default-free zero coupon bonds embedded in expression (5) above. These correspond to the last term in expression (9). Discounting the zero-recovery zero-coupon bonds enables similar liquidity discount impacts across different coupon bonds issued by the same credit entity.

Given this procedure, we can rewrite the coupon bond's value as

$$\begin{aligned} v_t^{liq} = \; & Lp(t,T)e^{\alpha_t(T-t)}\left[1 - Q(t,T)\right] \\ & + (C - L\delta_t)\textstyle\sum_{k=1}^{m} p(t,t_k)e^{\alpha_t(t_k-t)}\left[1 - Q(t,t_k)\right] \\ & + L\delta_t \textstyle\sum_{k=1}^{m} p(t,t_k)\left[1 - Q(t,t_{k-1})\right] \end{aligned} \tag{11}$$

where α_t is $\mathcal{F}_t$ - measurable. The $\mathcal{F}_t$ measurability of the liquidity parameter α_t implies that the market liquidity for any particular coupon bond is random. It is important to emphasize that in the subsequent estimation, both the recovery rate δ_t and the liquidity parameter α_t are stochastic, hence, they can vary randomly across time due to changing market conditions. Our estimation procedure allows for these estimated parameter values to reflect this randomness.[11] This is the valuation model estimated in the next section.

It is important to note that transactions costs (including bid/ask spreads) are a special case of a liquidity cost paid when trading, which are implicitly included via a liquidity discount (see Cetin, Jarrow and Protter [9] for the theoretical justification of this statement). Similarly, taxes paid on coupons and capital gains can also be interpreted as a type of transaction cost, and hence they are implicitly included in the liquidity discount as well.[12]

3.5 Comparative Statics - Misspecification Errors

In this section we build intuition for the misspecification errors when using the full-coupon recovery model, expression (10) instead of the reduced form (zero coupon recovery) model, expression (9). Recall that the misspecification error, the difference between the full-coupon recovery and the reduced form models' prices, is equal to $\sum_{k=1}^{m} C(m+1-k)\delta_t p(t,t_k)[Q(t,t_k) - Q(t,t_{k-1})]$. Note that, as defined, the misspecification errors are always positive. We next quantify the magnitudes of the misspecification errors and we provide a simple approximation that allows us to relate the misspecification errors to the model's inputs. Later, we relate the predicted misspecification errors to patterns in the data.

Misspecification Error Determinants: Calibration and Approximation

For illustrative purposes we make the following simplifying assumptions: (1) coupon bonds are priced on coupon dates, (2) the risk-free term structure of interest rates and the term structure of default probabilities are flat[13], (3) the coupon is set so that the zero coupon recovery model's bond price is equal to par, and (4)

[11] This is because we use implicit estimation at a fixed time t, and then we repeat the estimation under a different $\mathcal{F}_t$ at subsequent times.

[12] The complication of explicitly including liquidity costs (transaction, taxes) into the model is that different traders face different taxes and transaction costs based on their trading activities. Consequently, to determine a market price, an equilibrium model is needed. Equilibrium models are notoriously ladened with unrealistic assumptions. Furthermore, an argument can be made that the marginal trader, who determines the market price, is the lowest liquidity cost trader. Here, we note that many institutions pay small transaction costs and there do exist non-taxable institutions that purchase corporate debt.

[13] The assumption of a flat term structure is for illustrative purposes only. In the empirical implementation we use a full term structure of default probabilities, which is not assumed to be flat.

that there is no liquidity discount ($\alpha_t = 0$). This implies that the misspecification error is fully determined by the bond's maturity, the issuer's default probability, the recovery rate, and the risk-free rate. Table 1 reports misspecification errors for the different inputs, assuming that the risk-free term structure is flat at 2%. The par value of the bond is set to 100. As expected, misspecification errors increase with the bond's maturity, the issuer's default probability, and the recovery rate. For short maturity 2-year bonds, the misspecification errors are less than 0.10, while the misspecification errors are equal to or above 0.50 for the 10-year bonds. For 30-year bonds the misspecification errors are all larger than 3.61. The largest misspecification error is equal to 12.32 for a 30-year bond with a default probability of 2% and a recovery rate of 80%.

The misspecification error (above), which represents the present value of the recovery on all of the coupon payments after default, depends on the default probability, the recovery rate, the coupon payment, and the bond's maturity. For the first coupon payment the present value of the expected payoff in the event of default is equal to the discounted value of the product of the coupon rate, the recovery value, and the probability of default, so $C\delta_t p(t, t_1)Q(t, t_1)$. For the second coupon, default can occur either in the first or in the second period. The present value of the expected payoff in period one is the same as for the first coupon. The conditional expectation at time one of the expected payoff in period two is also the same as for the first coupon. However, this payment still needs to be discounted back to time zero and adjusted to take into account that the firm must have survived period one. For longer periods, a similar logic applies. It then follows that the misspecification error is exactly proportional to the product of the coupon payment and the recovery value. In contrast, the total misspecification error is only approximately proportional to the error resulting from the first coupon. The reason is that the probability of survival depends on the default probability and because payments farther into the future will be discounted by a larger amount.

Next we consider the effect of the bond's maturity or, equivalently, the number of coupon payments. Ignoring discounting and the adjustment for the probability of survival, the expected recovery payment from the second coupon is twice as large as for the first coupon. The reason is that default on the second coupon can happen either in the first or the second coupon period. For a bond receiving m coupon payments the factor is therefore $m(m+1)/2$. This means that the approximate total error is equal to $C\delta_t p(t, t_1)Q(t, t_1)m(m+1)/2$. We report this predicted error in the final column of Table 1. We also report the actual misspecification error, which is always smaller than the predicted error. The approximate error is very similar to the actual misspecification error, especially for short and intermediate maturities and when the default probability is small. For long maturity bonds and especially those with a high default probability and recovery rate the approximation is too large. Later, we will use these insights to analyze the pricing

Table 1: Full-Coupon Recovery Model Misspecification Errors

In this table we calculate prices based on the reduced form and the coupon recovery models. We assume a flat risk-free term structure of 2%, a flat default probability term structure and different maturities and recovery rates. Coupons are chosen so that bonds trade at par. We report the misspecification error resulting from using the coupon recovery instead of the reduced form model (column five, "missspec. error"). As discussed in the text, the misspecification error is approximately equal to C*PD*rec*(1/(1+rf))*m*(m+1)/2, which we report in the final column.

maturity	recovery	default probability	coupon	misspec. error	predicted misspec. error
2	0.4	0.01	2.61%	0.03	0.03
2	0.4	0.02	3.23%	0.06	0.06
2	0.8	0.01	2.21%	0.04	0.04
2	0.8	0.02	2.42%	0.09	0.10
5	0.4	0.01	2.61%	0.14	0.14
5	0.4	0.02	3.23%	0.33	0.35
5	0.8	0.01	2.21%	0.23	0.24
5	0.8	0.02	2.42%	0.50	0.53
10	0.4	0.01	2.61%	0.50	0.54
10	0.4	0.02	3.23%	1.19	1.34
10	0.8	0.01	2.21%	0.84	0.92
10	0.8	0.02	2.42%	1.78	2.02
30	0.4	0.01	2.61%	3.61	4.73
30	0.4	0.02	3.23%	8.21	11.71
30	0.8	0.01	2.21%	6.10	8.01
30	0.8	0.02	2.42%	12.32	17.57

errors observed in our empirical investigation. The approximate error serves as a reasonable predictor of the actual misspecification error expected to be observed in the data.

To summarize, the misspecification error is zero if the recovery rate, the default probability, or the coupon payment is zero. The error grows approximately with the square of the number of coupon payments and is exactly proportional to the product of the coupon payment and the recovery rate. Thus, bonds with significant recovery values, default probabilities, and with intermediate to long maturities will have significant misspecification errors.

Table 2: Different Spreads on Coupon and Principal

This table reports spreads appropriate for discounting coupons and principal (C and P) for various maturities, default probabilities, and recovery rates. As in the previous table, we assume a flat risk free term structure of 2% and a flat default probability term structure.

Def prob	0.01		0.02		0.01		0.02	
recovery	0.4		0.4		0.8		0.8	
Spread type	C	P	C	P	C	P	C	P
Maturity								
1	1.02%	0.61%	2.04%	1.21%	1.02%	0.20%	2.04%	0.40%
2	1.02%	0.60%	2.04%	1.20%	1.02%	0.19%	2.04%	0.38%
3	1.02%	0.59%	2.04%	1.19%	1.02%	0.18%	2.04%	0.36%
4	1.02%	0.59%	2.04%	1.17%	1.02%	0.17%	2.04%	0.34%
5	1.02%	0.58%	2.04%	1.16%	1.02%	0.16%	2.04%	0.32%
6	1.02%	0.58%	2.04%	1.14%	1.02%	0.15%	2.04%	0.30%
7	1.02%	0.57%	2.04%	1.13%	1.02%	0.14%	2.04%	0.28%
8	1.02%	0.57%	2.04%	1.12%	1.02%	0.13%	2.04%	0.26%
9	1.02%	0.56%	2.04%	1.10%	1.02%	0.12%	2.04%	0.24%
10	1.02%	0.55%	2.04%	1.09%	1.02%	0.11%	2.04%	0.22%

Pricing with Two Spread Curves

Pricing bonds using the same credit spread term structure is based on the incorrect assumption that coupons and principal have the same recovery. However, this does not mean that spreads cannot be used to price bonds. Instead, one needs to use two spread curves, one for discounting coupons and one for discounting the principal. If there is a misspecification error using the coupon recovery model to price bonds, then the two curves will be different.

Table 2 provides some illustrative examples of spread curves. We use the same methodology as before to illustrate these spreads, varying recovery rates and default probabilities as in the previous table. As long as there is positive recovery, coupon spreads lie above principal spreads since the latter will be worth more and thus are discounted by less. A higher default probability makes all spreads higher; in the example all the spreads are approximately proportional to the default probability. Comparing spreads, the difference for coupons and principal is close to the product of the default probability and the recovery rate, which follows from the misspecification error relationship we showed above, where, for the first coupon, the misspecification error is equal to $C\delta_t p(t,t_1)Q(t,t_1)$.

It is useful to note that the standard spread calculation, which assumes equal seniority of principal and coupons, will result in a spread that cannot be used to discount either cash flows with zero or positive recovery. For the former (the coupons), the spread will be too low and for the latter (the principal) it will be too high. Thus, using a single spread (or spread curve) to price a new bond with a different maturity or coupon will result in misspecification errors. In addition, using this 'standard' spread calculation to assess the market's implied risk pricing is not possible.

4 Estimation Procedures

The details of the estimation procedures are as follows. To fit the valuation model to market prices, we obtained traded coupon bond prices for the 454 trading days between September 1, 2017 and June 30, 2019 using the TRACE system. The price in the TRACE system does not represent the full amount paid for the bond by the buyer. The full amount paid is the price plus accrued interest. We compare the full amount paid (the present value of the bond purchase) with the valuation model in expression (11). For each firm, we eliminated any subordinated bonds, callable and putable bonds, structured bonds, bonds with "death puts" or a "survivor option," and floating rate bonds from the sample. Survivor option bonds distort bond prices both because they are issued in small amounts (typically $20 million or less per tranche) and because the value of the embedded put option is significant. The survivor option feature has become more common in recent years.[14] Finally, to be included in our sample, the bond issue's daily trade volume had to exceed $50,000 (in almost every case, volume was much larger) and with at least two bonds traded.[15] We also exclude some bonds of European issuers subject to a 2014 EU regulation allowing regulators to demand an exchange of senior debt securities into equity. Because data assembly and cleaning costs are substantial,[16] we restrict our attention to this sample.

We use the U.S. Treasury yields reported daily by the U.S. Department of the Treasury[17] and derive the maximum smoothness Treasury forward rate curves

[14] The largest issuers of survivor option bonds as of 2016 included General Electric, Goldman Sachs, Bank of America, Wells Fargo, Ford Motor, HSBC Holdings, National Rural Utilities Cooperative Finance Corporation, Dow Chemical, Prospect Capital, and Barclays PLC. A typical survivor option bond's terms are described as follows in a recent prospectus supplement from General Electric Capital Corporation: "Specific notes may contain a provision permitting the optional repayment of those notes prior to stated maturity, if requested by the authorized representative of the beneficial owner of those notes, following the death of the beneficial owner of the notes, so long as the notes were owned by the beneficial owner or his or her estate at least six months prior to the request. This feature is referred to as a "Survivor's Option." Your notes will not be repaid in this manner unless the pricing supplement for your notes provides for the Survivor's Option. The right to exercise the Survivor's Option is subject to limits set by us on (1) the permitted dollar amount of total exercises by all holders of notes in any calendar year, and (2) the permitted dollar amount of an individual exercise by a holder of a note in any calendar year."

[15] To ensure model convergence, for issuer-days with only two observations we also require that the maturities are more than 270 days apart.

[16] It is necessary to screen out callables and survivor options, data on which is only available in the pricing supplement. The SEC and FINRA do not maintain public access to prospectus data for more than about 5 years in easily accessible form. Thus including, for example, data from the financial crisis is not feasible. In addition, the TABB group finds a very high frequency of errors "TABB Group analysis shows reconciliation differences in more than 20% of new issues." (http://www.finregalert.com/an-sec-mandated-corporate-bond-data-monopoly-will-not-help-quality/). There are also non–trivial computational costs.

[17] https://www.treasury.gov/resource-center/data-chart-center/interest-

from these data (see Adams and van Deventer [1]). Using these historical forward rate curves, we compute the term structure of default-free zero coupon bond prices on all dates.

The default process parameters (θ, ϕ) from the proportional hazard rate model were provided by the Kamakura Risk Information Services division of Kamakura Corporation.[18] Kamakura uses a refinement of the approach employed by Chava and Jarrow [10] to estimate these parameters that are then used to construct the full term structure of cumulative default probabilities.[19] Specifically, for each issuer-day we calculate cumulative default probabilities from the 10-year term structure of monthly marginal default probabilities (the monthly probability of default conditional on no prior default) that Kamakura generates from separate maturity-specific models. The state variables used in the Kamakura hazard rate estimation include both firm specific and macroeconomic variables.

Given the zero coupon bond prices and the default process (above), we estimate the recovery rate δ_t and liquidity discount factor α_t for each issuer on each day. We compared the model values (expression (11)) to the market prices using a non-linear least squares estimation, calculated on a trade-volume weighted basis, to solve for the best fitting values of $(\delta_t,\, \alpha_t)$. To compute expression (11), we discretized time. Then, we used the parameter estimates obtained above to simulate and to evaluate the expectation in expression (11).[20]

5 Illustrations

Before moving to the full sample model estimation, this section provides evidence that market participants are aware of the difference in seniority between principal and coupons in default. We consider three companies that filed for bankruptcy: Lehman, PG&E, and Weatherford International. Lehman is chosen because of the size and importance of its bankruptcy. The latter two firms are in our sample because each firm has a sufficient number of trades on the trade date. In each case we focus on senior bonds, including callable bonds, because on the day bankruptcy is announced the call option is worthless and can be ignored. We fit the reduced form and full-coupon recovery models to the data. A useful feature of analyzing issuer bonds after they file for bankruptcy is that market participants agree that the default probability equals 100%. The recovery amount for the zero coupon

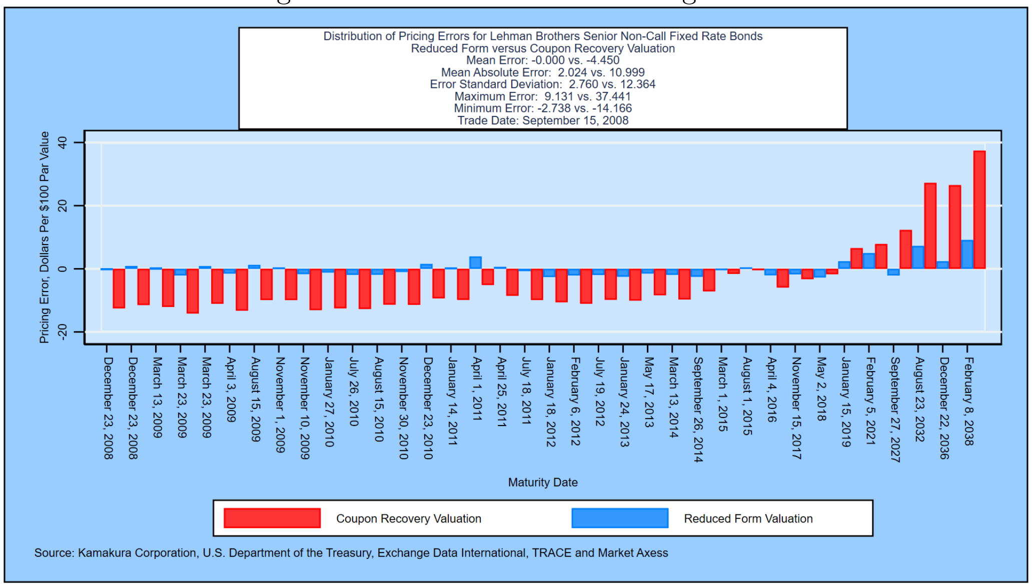
rates/Pages/TextView.aspx?data=yield

[18]See www.kamakuraco.com.

[19]The model underlying the default probability calculations is similar to the one used in Campbell, Hilscher and Szilagyi [7, 8], who extend Chava and Jarrow [7] and Shumway [36]. Campbell et al. show that the default probability measure is a more accurate predictor of failure than Moody's EDF numbers, data that have been widely used in academic studies, e.g. Berndt, Douglas, Duffie and Ferguson [4].

[20]Of course, the estimation of $(\delta_t,\, \alpha_t)$ was done simultaneously with this computation.

Figure 1: Lehman Brothers Pricing Errors



recovery model is the number of recovery units times a notional amount of \$100 (par value) for each of the bonds. The recovery amount for the full-coupon recovery model is \$100 plus the dollar coupon times the number of remaining payments on each bond, a different amount for each issue. For each issuer day and for both of the models, we ran the regression NPV = (delta)(notional recovery amount) to derive the recovery rate parameter and the net present values (price plus accrued interest) for each bond.

Figures 1-3 depict the pricing errors. We order the bonds by maturity. Pricing errors when using the reduced form model (in blue) are substantially lower than those resulting from the full-coupon recovery model (in red). Mean absolute errors are more than five times as large for Lehman (2.0 vs. 11.0) and almost ten times as large for PG&E (2.1 vs. 19.7) and Weatherford International (2.6 vs. 21.0). Running regressions of actual versus predicted prices, for all three firms the R^2 's are larger than 99% for the reduced form model. When using the coupon recovery model, the R^2 's are between 84% and 92%.

The full-coupon recovery model results in prices that are too large, especially for bonds of longer maturities that have more coupons, which, if there were of equal seniority, would entitle the bond holder to a recovery value. However, in default those coupons are worthless and so any coupon paying bond would have pricing errors that are positive as long as the model was using unbiased inputs. However, in an attempt to fit the data, the model tries to reduce the average pricing error resulting in bonds with short maturities being underpriced and bonds with long

Figure 2: Pacific Gas & Electric Pricing Errors

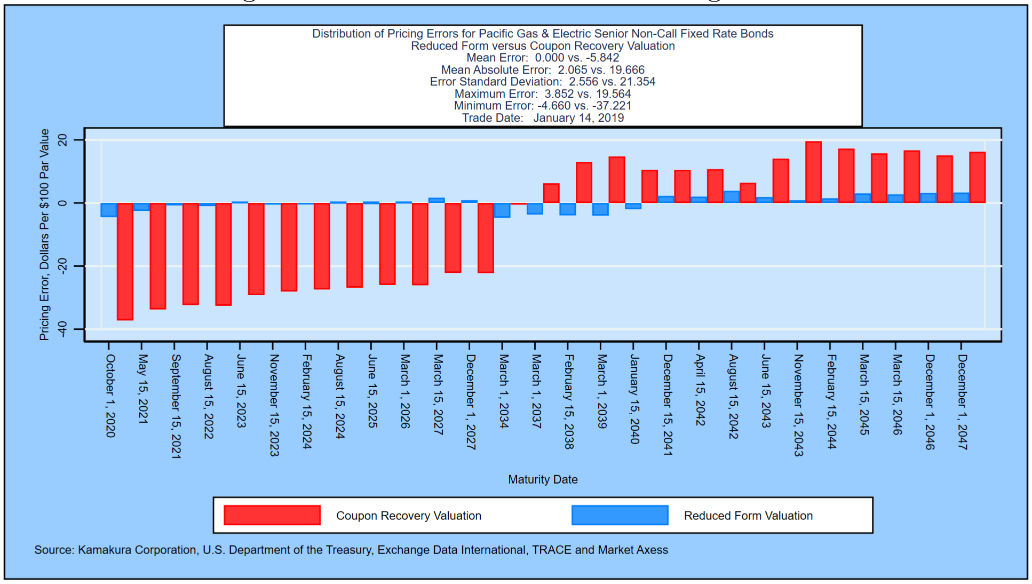
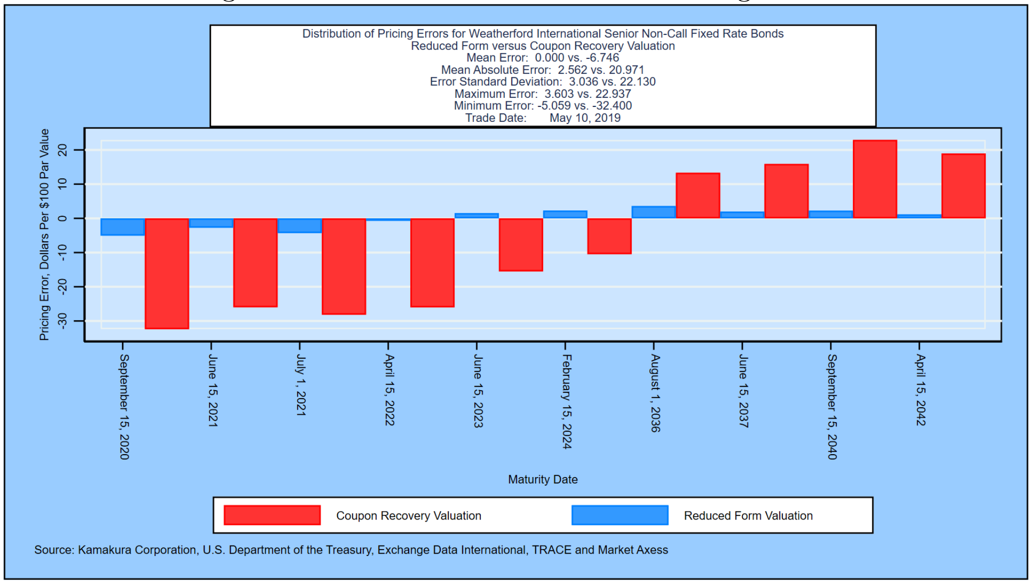


Figure 3: Weatherford International Pricing Errors



maturities being overpriced. The maximum errors lie between 19.6 and 37.4. It is worth noting that average market prices are equal to 32.4 (Lehman), 78.2 (PG&E), and 65.0 (Weatherford International) so that the maximum errors are around one half the market price. The (negative) minimum errors are similar in size lying between -37.2 and -14.2. In contrast, the reduced form maximum and minimum pricing errors are much smaller. They lie between 3.9 and 9.1 and -5 and -2.7, and so are approximately one quarter of the full coupon recovery model pricing errors. Importantly, and in direct support of the reduced form model, reduced form model pricing errors have no clear pattern relative to the bond's maturity.

To summarize, Lehman, PG&E and Weatherford International's bond prices provide direct evidence in support of the reduced form model relative to the full-coupon recovery model. Failing to take into account the different seniority of coupons and principal results in substantial pricing errors, which have a predictable pattern in line with our model.

6 A Comparative Analysis

This and the next section provide a comparative analysis of the reduced form valuation model versus two alternatives. One is the full-coupon recovery model, which is the traditional recovery rate model that assumes recovery on unpaid coupons contained in expression (10). In section 3.4 we discussed determinants of the misspecification errors for this model and we now provide a full-sample analysis in which we calculate the model's misspecification errors and compare pricing errors across both models. The second model is a ratings-based valuation model, which is a special case of the full-coupon recovery model. The ratings-based valuation model is consistent with numerous pronouncements from the Basel Committee on Banking Supervision [2, 3]. The assumption is that the credit spread depends only on the rating. Relative to the full-coupon recovery model, which restricts the spread to be the same for coupons and principal, the ratings model also assumes a constant spread across maturities and for all issuers with the same rating. To obtain the ratings-based valuation, employing all bonds of each ratings grade, we used a non-linear least squares estimation procedure to solve for the credit spread which, when combined with the matched-maturity U.S. Treasury zero coupon bond prices, produced the minimum sum of squared errors in fitting the model's net present value (price plus accrued interest) using expression (10) to market prices. It is important to emphasize that by utilizing expression (10) the ratings based approach to valuation ignores the different recovery rates on the coupon payments received before and after default (and as discussed in Section 3 above).

After the restrictions discussed above (dropping callable bonds etc.) and having estimated recovery and liquidity parameters, our sample consists of a little more than 70 thousand observations for 151 issuers. In each case we observe the actual (invoice) price of the bond as well as the reduced form (zero coupon recovery) model price. In addition, we have calculated the full-coupon recovery and the ratings model prices. The full-coupon recovery model produces different prices only if predicted misspecification errors (see section 3.4) are non-zero. We therefore restrict attention to bonds with at least one coupon outstanding (maturity larger than six months), a recovery value of at least 1%, a coupon rate equal to or greater than 1%, and at least two observations on each issuer day satisfying these restrictions. These are the observations for which we know that the full coupon recovery model and the reduced form model have different prices. After these restrictions, 49.4 thousand observations for 112 issuers remain. We perform a comprehensive analysis of the reduced form model's pricing errors in section 7.

Table 3 reports summary statistics. The median coupon rate is 2.9%, the median maturity is 2.7 years, the median default probability is 0.6%, the median recovery rate is 18%, and the median liquidity parameter is -0.1%. The first three are inputs and the latter two are estimated using the reduced form model. The median rating is A-.[21] Using these estimates as inputs we next calculate prices using the full-coupon recovery instead of the reduced form model. This calculation of the misspecification error is the error that would be obtained if someone had access to the parameters of the model but was using the incorrect model (which assumes non-zero recovery of coupons). Results are in the second to last column of the table. The median misspecification error is nine cents, 25% of the data has errors larger than \$0.28, and for 5% of the data the misspecification error is larger than \$1.43. The median observed net present value is equal to 100.37. As a result, these numbers are directly comparable to those in Table 1 discussed in Section 3.4., where we analyzed predicted misspecification errors calculated using an approximation. Indeed, when we regress actual on predicted misspecification errors, the coefficient is indistinguishable from one and the R^2 is 71%.

These numbers are important and informative when pricing bonds. Consider the following two examples. First, an issuer or bank intending to underwrite or buy a new issue may want to price bonds prior to issuance. What we show is that knowledge of the correct inputs, but using an incorrect model, can result in large misspecification errors. Second, suppose that an investor observes a set of prices that reflects the differences in seniority of principal and coupons today, but wants to predict future bond prices or to price other bonds that are not traded. Again, using the incorrect model can result in substantial errors.

Next, we compare the two models when fitting them to the data independently. For each issuer-day, we estimate both the reduced form and the full-coupon recov-

[21] The sample consists primarily of investment grade bonds since many high yield bonds have call features, all of which are excluded. An analysis of callable bonds goes beyond the scope of this paper.

Table 3: Pricing Error Summary Statistics

This table reports summary statistics for the sample we use to analyze pricing errors from using the full coupon recovery model instead of the reduced form model. The first three columns report the model inputs coupon, maturity, and default probability; we also report the credit rating for reference. Columns five and six report statistics for fitted recovery rates and the liquidity parameter alpha. The column 'Misspec. error' reports the misspecification error, the difference between the coupon recovery model and the reduced form model when both are calculated based on the unbiased reduced form model parameters. The column 'Error diff' reports the difference in the absolute value of the pricing error (model price minus observed price) using the full coupon recovery model instead of the reduced form model. The sample consists of all observations with a recovery rate larger or equal to 1%, a maturity larger or equal to 6 months, a coupon rate larger or equal to 1%, and at least two issuer-day observations.

	Coupon	Maturity	Def Prob	Rating	Recovery	Liquidity	Misspec. Error	Error difference
p5	1.8	0.8	0.1	AA-	0.05	-0.7%	0.00	-0.03
p25	2.5	1.7	0.2	A+	0.12	-0.4%	0.03	0.00
p50	2.9	2.7	0.6	A-	0.18	-0.1%	0.09	0.00
p75	3.8	4.4	1.2	BBB+	0.35	0.0%	0.28	0.03
p95	5.8	8.1	3.9	BBB-	0.64	0.0%	1.43	0.33
Mean	3.2	3.3	1.0		0.25	-0.2%	0.29	0.05
SD	1.2	2.2	1.3		0.19	0.3%	0.53	0.26
Number of observations: 49,415								

ery models. We require that there be at least four observations each issuer-day. This restriction is imposed because there are two parameters to estimate and including more observations decreases the flexibility of the model.[22] We then calculate the difference between the absolute values of the pricing errors from the coupon recovery and the reduced form model. This error difference is reported in the final column of Table 3. Due to the restrictions imposed on coupons, the recovery rate, and the bond's maturity, the two models differ and so the pricing error differences are non-zero.

Table 4 explores the determinants of the error differences. On average the reduced form model does a better job (also see the final column of Table 3). Pricing errors are 4.8 cents larger with the full-coupon recovery model and the difference is statistically significant. We expect that the (in)ability of the full-coupon recovery model to fit the data reflects its misspecified assumption. Recall that when estimating this model, the algorithm searches for those parameters to maximize its fit.[23] When predicted misspecification errors are small, for example

[22]Following the specification of our model, for each day we estimate the recovery rate and the liquidity parameters. If we had made additional assumptions, for example that the liquidity parameter and the recovery rate are approximately constant over some time span, we could have increased the numbers of observations relative to the parameters. We would expect this change to result in the reduced form model outperformance being even larger. However, as discussed earlier, our model allows these parameters to change on a daily basis and so we do not pool observations into larger groups.

[23]The examples of the issuers in bankruptcy nicely show that considering only the average

Table 4: Regressions of Pricing Error Differences

This table reports results from regressions of pricing error differences (between the absolute value of the pricing error in the full coupon recovery model and the absolute value of the reduced form model) in Panel A and mean-square error differences and R-square differences in Panel B. Pricing error differences are calculated for the sample from Table 5 with the additional restriction that the firm-day has at least four observations. MSE (columns one to four) and R2 (columns five to eight) differences are based on firm-day regressions of prices on predicted prices. MSE difference is the difference between the coupon recovery and the reduced form model mean squared error. For R-square it is the other way around. "Misspec error SD" is the within firm-day standard deviation of the predicted misspecification error. High PD and "high mis err SD" are the top quartiles of the data with the highest default probability and misspecification error SD respectively. Standard errors, reported below coefficients, are clustered by issuer and date; *** denotes significant at the 1% level, ** at the 5% level and * at the 10%

Panel A: Error difference (of the absolute values of full coupon and reduced from model pricing errors)								
Recovery				0.179**	-0.034			
				(0.087)	(0.021)			
Maturity				0.005	0.001			
				(0.004)	(0.003)			
Coupon				-0.004	-0.005			
				(0.005)	(0.004)			
Def Prob				0.049***	0.017***			
				(0.003)	(0.003)			
Misspec error SD					0.294***	0.341***		
					(0.009)	(0.014)		
Constant	0.048*	0.180***	0.179***	-0.047**	0.004	-0.007***		
	(0.025)	(0.034)	(0.035)	(0.020)	(0.007)	(0.002)		
Sample	full	high PD	high mis err SD	full	full	full		
R-square				0.2647	0.3208	0.3154		
Observations	38,291	9,569	9,564	38,291	38,291	38,291		
Panel B: MSE and R2 differences								
	MSE difference				R-square difference			
Misspec error SD				0.198***				0.068***
				(0.045)				(0.015)
Constant	0.027*	0.104***	0.103***	-0.000	0.009**	0.034***	0.033***	-0.000
	(0.014)	(0.024)	(0.027)	(0.004)	(0.004)	(0.008)	(0.009)	(0.001)
Sample	full	high PD	high mis err SD	full	full	high PD	high mis err SD	full
R-square				0.4353				0.3645
Observations	5,646	1,411	1,411	5,646	5,646	1,411	1,411	5,646

because the default probabilities are low, there is only a small difference between the two models and the pricing error differences will also be small. But even if predicted misspecification errors are large, the full-coupon recovery model may still generate lower pricing errors, for example, if the bonds in the specific firm-day sample have very similar predicted misspecification errors. In this case the misspecified model will be able to adjust, for example by choosing a lower recovery rate, and the resulting pricing error differences relative to the reduced form model will be low, albeit at the cost of biased model parameter estimates. However, if bonds in the issuer-day sample have very different predicted misspecification errors, for example because of a large dispersion in maturity combined with large coupons and significant default probabilities, the full-coupon recovery model will find it much harder to price the bonds in the sample. Here, the misspecified model does not have sufficient degrees of freedom to match the data well. We find evidence that supports both of these conjectures. When restricting attention to observations in the top quartile of default probabilities, the full coupon recovery model's pricing error is 18 cents larger on average. The average pricing error difference is almost the same if we focus on the quartile of observations that should be the most difficult for the full-coupon recovery model to fit: those observations

pricing error of the full coupon recovery model misses the strong association of the pricing errors with the bond's maturity, a pattern that is a direct consequence of the model's misspecification.

with the highest issuer-day predicted misspecification error standard deviation.

Pricing error differences are large when the model predicts them to be large. We next regress pricing error differences on the four variables that are the main determinants of the misspecification error (recovery rate, maturity, coupon, and default probability). The recovery rate and the default probability are highly significant and come in with a positive sign; the maturity and coupon are insignificant. Of course, these variables do not measure the dispersion of the misspecification errors, which we expect to be an important determinant of the reduced form model's outperformance. When we add the misspecification error standard deviation together with the four variables, dispersion is the most significant variable. The recovery rate is no longer significant and the coefficient on the default probability is cut in half. More telling, including only the misspecification error standard deviation explains almost the same amount of variation as compared to including the four input variables. The R^2 is 31.5% compared to 32.1%.

Next, for each of the 5,646 issuer days we regress observed prices on predicted prices. We do this using both the reduced form and the full-coupon recovery models. We calculate the mean squared error and the R^2 differences across the two models. The results are reported in Table 4 Panel B. The reduced form model has, on average, a 0.027 lower mean squared error than the full-coupon recovery model. We again find that the reduced form model's outperformance is much larger when the default probability is high. The MSE difference is close to four times as large (0.104) for the highest quartile of default probability observations. It is also very similar if we instead restrict attention to the top 25% of the predicted misspecification error standard deviation issuer days. When the misspecification error standard deviation is large, the MSE difference is high – the regression R^2 is 43.5%. Those are the times when we expect the full-coupon recovery model to have the most difficulty in fitting the data. The R^2 differences between the two models are also high when the default probabilities and the misspecification error standard deviations are high. Relative to an average of 0.9%, the R^2 differences more than triples to 3.3% and 3.4% in the difficult-to-price subsamples. The error standard deviation explains 36.5% of the variation in the R^2 differences.

To summarize, this analysis provides evidence that the pricing error differences between the two models are statistically significant, sometimes substantial, and that the misspecified full-coupon recovery model has a much more difficult time fitting the data when our model predicts it will. This evidence is for a sample consisting of 96% investment grade debt (99.7% with a rating of BB+ or above) and thus one where market participants perceive default is not imminent. As we discussed earlier, when default has happened, differences across the models are even higher.

7 Times Series Comparison

We now examine the performance of the models, including the ratings-based model, over time. Our sample included a wide variety of market conditions from both a liquidity and volume perspective because of the Christmas, New Year's, Martin Luther King's birthday and other holidays in the United States.

Figure 4 compares the R^2 of a regression of prices on predicted prices between the three valuation models. The evidence in this and the following figures makes it immediately apparent that the ratings-based model has by far the worst fit. The R^2 for the reduced form valuation methodology, shown in blue, was higher than the ratings based model's R^2, shown in red, on almost every trading day. The R^2 advantage over the full-coupon recovery model was small in the first half of the sample, but with exceptions, and then increased.

Figure 4: R^2 Statistics

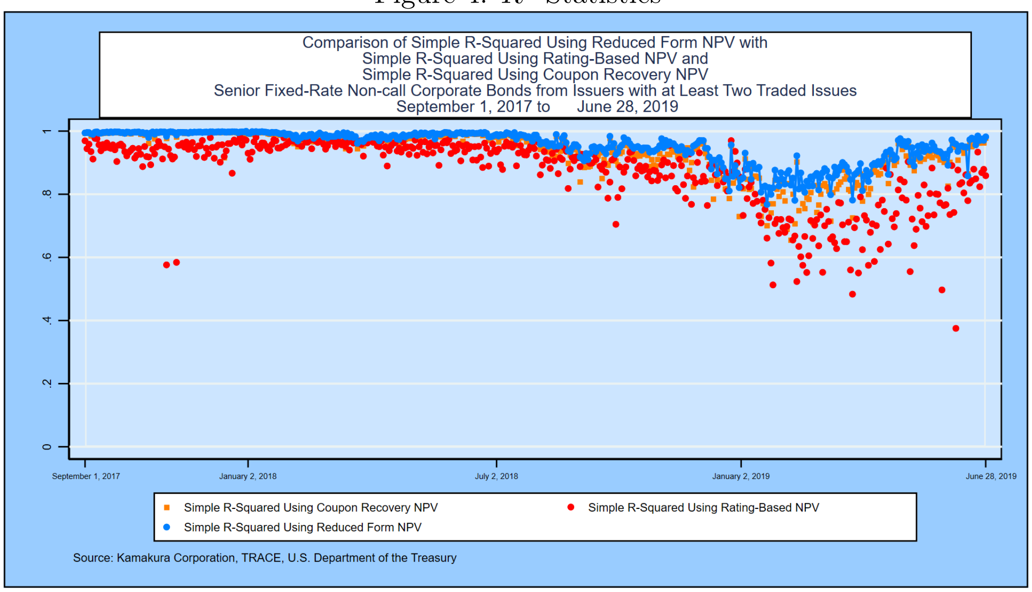
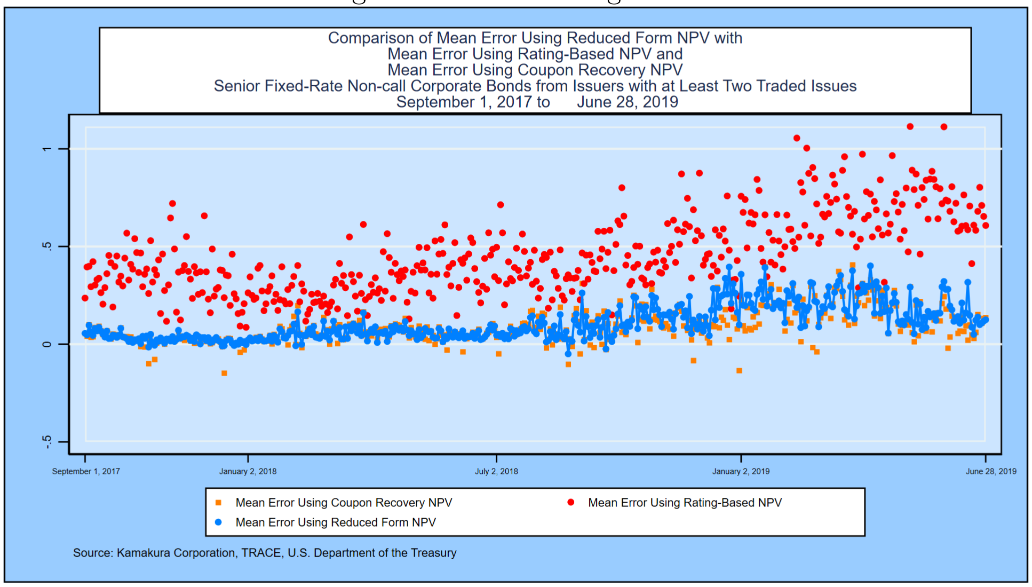


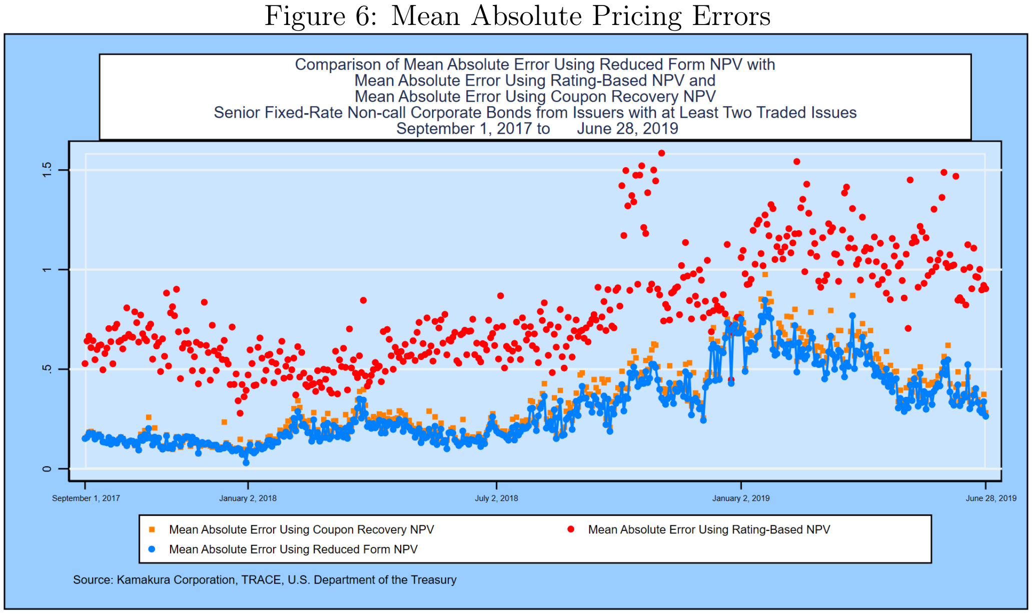
Figure 5 graphs the mean pricing errors, which show that on most trading days, the mean pricing error was closer to zero for the reduced form valuation approach. Pricing errors under the ratings-based valuation approach are biased high because of highly skewed credit risk conditions among companies with the same credit rating. The advantage of the reduced form valuation model over the full-coupon recovery model was significant while being smaller than the advantage relative to the ratings-based model.

Figure 5: Mean Pricing Errors



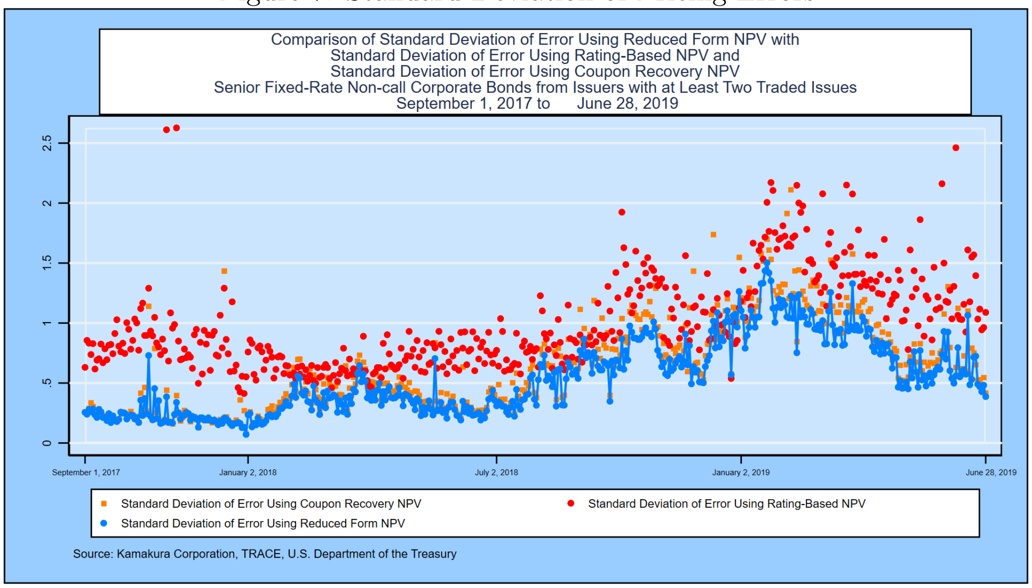
The mean absolute pricing errors are provided in Figure 6. As depicted, the mean absolute error was consistently largest for the ratings-based model and smallest for the reduced form model, which also outperformed the full-coupon model using this measure.

Figure 6: Mean Absolute Pricing Errors



Lastly, Figure 7 provides the time series of the standard deviation of the pricing errors. The pattern in error standard deviations is similar to those in mean absolute error: the reduced form model had the best fit and the ratings-based model the worst fit.

Figure 7: Standard Deviation of Pricing Errors



As evidenced by these comparisons, the reduced form valuation methodology provides substantially more accurate valuations than does the rating-based methodology based on expression (10). The advantage over the full-coupon recovery model is smaller but consistent. As we have shown above, the difference between the full-coupon and reduced form models becomes larger for firms in bankruptcy or when default probabilities are high.

8 Specification Tests

This section performs specification tests of the model's pricing errors to better understand its goodness of fit to market prices. We do not find large and consistent patterns in the pricing errors and so we conclude that the reduced form model is well specified. The details of this exercise are as follows. We ran two regressions with various independent variables on both the magnitude of pricing errors and their absolute values. We seek to explain the sources of the pricing errors for the full sample of 70.6 thousand observations on 454 trading days for 151 issuers. The average pricing error over the full sample is minus \$0.108 for bonds with a par value of \$100. The standard deviation of the pricing errors is \$0.625. The results

Table 5: Regressions on Reduced Form Pricing Errors

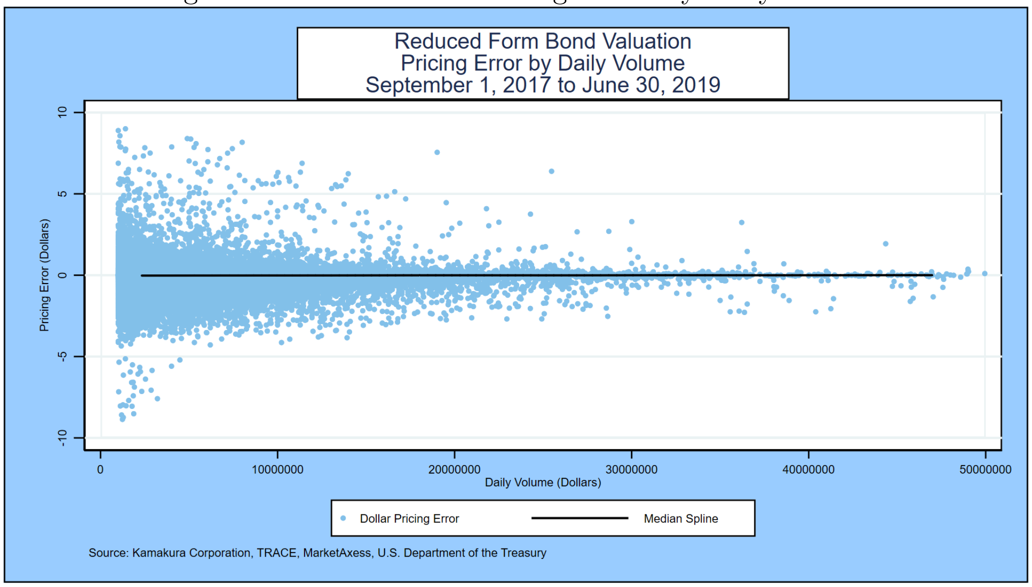
This table reports results from regressing reduced form model pricing error and absolute pricing error on explanatory variables. Gross premium and discount are equal to the premium or discount of the bond price and zero otherwise. Trading volume is measured in millions. Ford and Morgan Stanley are issuer-specific dummy variables. Both regressions include rating dummies. Standard errors, which are reported below the coefficients, are clustered by date and issuer. *** denotes significant at the 1% level, ** at the 5% level and * at the 10% level.

	Error	Absolute error
maturity	0.046***	0.026***
	(0.010)	(0.005)
GrossPremium	-0.001	0.011*
	(0.008)	(0.007)
AnnualizedPremium	0.002**	-0.001**
	(0.001)	(0.001)
GrossDiscount	0.221*	0.084
	(0.126)	(0.054)
AnnualizedDiscount	-0.643**	0.033
	(0.323)	(0.058)
Volume	0.003***	-0.005***
	(0.001)	(0.001)
Volume*MorganStanley	-0.002***	0.002*
	(0.001)	(0.001)
Ford	-0.234**	0.391*
	(0.110)	(0.222)
MorganStanley	0.071**	-0.082***
	(0.031)	(0.023)
Observations	70,586	70,586
R-squared	0.1753	0.2854

of the regression are shown in Table 5.

The regression in Table 5 explains 17.5% of the variation in pricing errors. Dummy variables for two issuers are statistically significant in predicting pricing errors: Morgan Stanley and Ford. Daily trading volume is also statistically significant but with a coefficient that is small. However, trading volume is somewhat important for Morgan Stanley bonds with a coefficient of -0.2 cents (trading volume is measured in millions). Years to maturity is statistically significant in predicting the pricing error, with the errors increasing by 4.6 cents for each additional year to maturity. The gross premium and annualized premium (gross premium divided by years to maturity) have small or statistically insignificant effects. In contrast, the gross discount and annualized discount are more important

Figure 8: Reduced Form Pricing Errors by Daily Volume



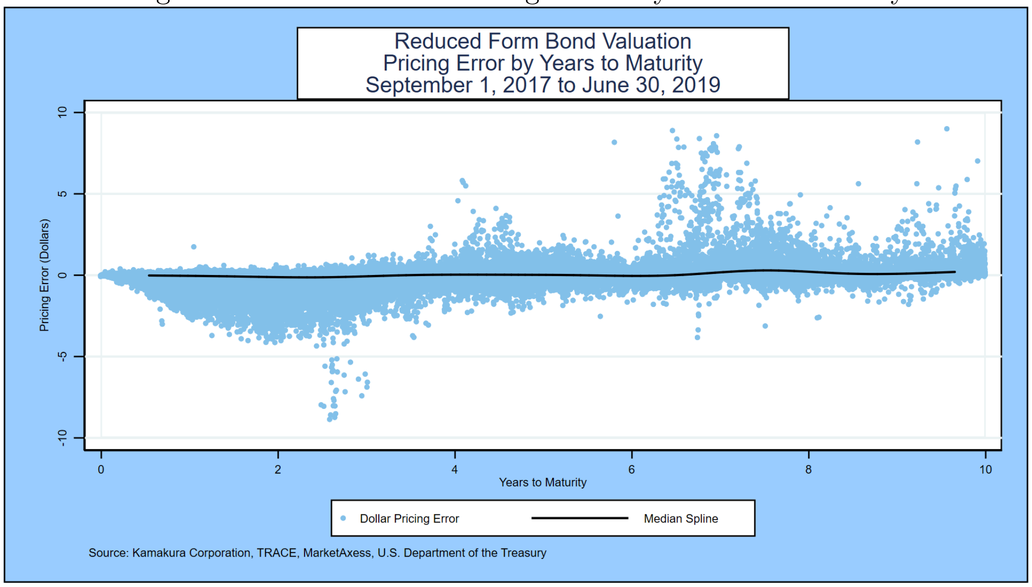
for the pricing error, though not the absolute error. The root mean squared error is \$0.568, 6 cents less than the simple standard deviation of the pricing error. In summary, the impact of the predictive variables is small in magnitude.

Table 5 also contains the regression that explains the pricing error volatility by fitting the absolute value of the pricing errors. The mean absolute pricing error is \$0.31 over the full sample with a standard deviation of mean absolute error equal to \$0.56. The results show that the dummy variables for the two firms impact the absolute value of pricing errors: Morgan Stanley and Ford. The coefficient of the daily volume is again significant but fairly small; the interaction with Morgan Stanley is equal to -0.2 cents. In terms of premiums and discounts, the absolute pricing error increases with the gross premium and annualized premium, though the former effect is significant only at the 10% level. The root mean squared error is \$0.47, just 9 cents better than the raw standard deviation of the absolute value of pricing error.

To visually illustrate the impact of the independent variables on the pricing errors, Figure 8 shows that the model's pricing error exhibits greater volatility for bonds whose daily trading volume is well under \$10 million dollars. This is consistent with the experience of market participants.

In Figure 9, we plot the pricing errors as a function of remaining years to maturity. Our sample consists solely of bonds with a remaining maturity of 10 years or less because the longest maturity default probabilities available were 10-year default probabilities. The volatility of the pricing errors seems to increase

Figure 9: Reduced Form Pricing Errors by Years to Maturity

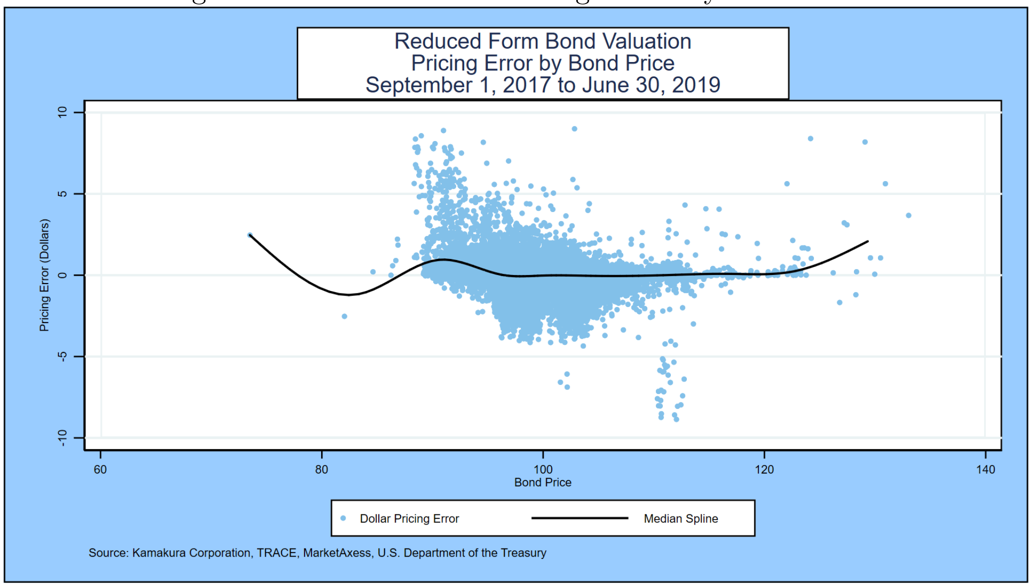


with maturity but the median spline exhibits no large deviation from zero. Finally, we examine the possibility of tax effects for bonds with large price premiums or discounts. Figure 10 plots pricing errors as a function of the true market price of the bonds. Indeed, the median spline moves upward for very high premiums over par.

9 Conclusion

This paper derives a new and tractable coupon bond valuation model, which includes a more realistic recovery rate process that distinguishes between coupon payments received before and after default. Our approach has several benefits: (1) we show how to price bonds as portfolios of zero recovery zero coupon bonds, which pay off in the event of no default, and digital recovery bonds, which pay off in the event of default. Using this insight, the model is intuitive and straightforward to implement empirically. (2) Bond prices depend on only four inputs: the risk-free term structure, the default probability term structure, the recovery rate, and a parameter capturing liquidity. (3) We provide a simple approximation to misspecification errors resulting from using the full-coupon recovery model rather than the reduced form model. Misspecification errors depend directly on the coupon, recovery rate, default probability, and time to maturity, and they can be substantial in size. (4) We show how the common practice of pricing bonds using credit spread curves can be adapted to using two spread curves, one for

Figure 10: Reduced Form Pricing Errors by Bond Price



coupons and one for the payment of principal.

Using a large data set of corporate coupon bond transaction prices we illustrate the application of our model to market prices. We find that our model fits the data well. The pricing errors between the model and market prices are small. We find clear evidence that the reduced form model outperforms the full-coupon recovery model, confirming the need for taking into account zero recovery on coupons. Outperformance is large in particular when default probabilities are high and when we expect the misspecified full-coupon recovery model having insufficient degrees of freedom to be able to fit observed bond prices. Model outperformance is immediately evident when considering bond prices of companies in bankruptcy. We also find that the reduced form valuation approach is much more accurate versus a traditional ratings-based valuation methodology, which underlies the capital regulations from the Basel Committee on Banking Supervision.

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