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Reduced-Form Valuation of Callable Corporate Bonds: Theory and Evidence

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Reduced-Form Valuation of Callable Corporate Bonds: Theory and Evidence

Abstract

We develop a reduced-form approach for valuing callable corporate bonds by characterizing the call probability via an intensity process. Asymmetric information and market frictions justify the existence of a call-arrival intensity from the market’s perspective. Our approach both extends the reduced-form model of Duffie and Singleton (1999) for defaultable bonds to callable bonds and captures some important differences between call and default decisions. We also provide one of the first comprehensive empirical analyses of callable bonds using both our model and the more traditional American option approach for valuing callable bonds. Our empirical results show that the reduced-form model fits callable bond prices well and that it outperforms the traditional approach both in- and out-of-sample.

JEL Classification: C4, C5, G1
1 Introduction

One of the most exciting developments in the fixed-income literature in the last decade is the reduced-form model for corporate bonds pioneered by Jarrow and Turnbull (1995) and Duffie and Singleton (1999). By modeling the default probability as an intensity process, this approach greatly simplifies the valuation of corporate bonds and makes large scale empirical analysis of corporate bonds possible. In addition to corporate bonds, empirical studies have applied this approach to a wide variety of securities with default risk, such as sovereign bonds, interest rate swaps, and credit derivatives.\footnote{Dai and Singleton (2003) provide a survey of the empirical literature on dynamic term structure models for default-free and defaultable bonds.}

A striking feature of this literature is that most empirical studies have focused only on non-callable bonds, even though as pointed out by Duffie and Singleton (1999), “the majority of dollar-denominated corporate bonds are callable.”\footnote{In the Lehman Brothers Fixed Income Database, there are in total 5,009 firms and 32,295 bonds from 1973 to 1996. We find 12,509 callable and 8,888 non-callable bonds with no other provisions.} The dominant reason for this omission is that callable bonds are more difficult to price and, at present, there exist no accurate and tractable valuation methods suitable for a large scale empirical analysis of callable corporate bonds.

Indeed, the traditional structural approach for valuing callable corporate bonds typically assumes a stochastic process for firm value and determines the optimal call policy by minimizing the present value of its liabilities. Determining this optimal call policy in the presence of default risk and possibly other callable bonds is a challenging task. It requires knowledge of both the firm’s value process and the firm’s dynamic liability structure, which are typically unobservable to market participants. Furthermore, it is difficult to fully incorporate the impact of market frictions, which can significantly change the optimal call policy.\footnote{See Ingersoll (1977) for the justification of this assertion for convertible bonds.} And, even given all of these elements, valuation itself is an enormous computational exercise.\footnote{See Jones, Mason, Rosenfeld (1984) and Berndt (2004).} Therefore, while the structural approach provides useful insights on the valuation of callable bonds (see Acharya and Carpenter (2002)), it is very difficult to implement in practice.

Given the limitations of the structural approach, Duffie and Singleton (1999) propose to value callable corporate bonds as American options written on otherwise identical non-callable bonds valued using the reduced-form approach. Although this does not require information on the firm’s assets and capital structure, it assumes that the firm calls the bond to minimize the market value of the particular bond under analysis. This assumption ignores this bond’s impact on the firm’s remaining liabilities, and hence shareholder’s equity may not be maximized. The optimal call policy determination also ignores market frictions. Finally, the American feature is
computationally difficult because valuing American interest rate options is numerically intensive. As a result, empirical implementations of this approach tend to be of limited scale. For example, in one of the few applications of this approach, Berndt (2004) only considers one firm, Occidental Petroleum.

The objectives of our paper are: (i) to develop a valuation method for callable corporate bonds that is as tractable as the reduced-form model for credit risk, yet still captures the relevant differences between call and default decisions; and (ii) to apply the new method to conduct a comprehensive empirical pricing analysis of callable corporate bonds. Thus, the contributions of our paper are both methodological and empirical.

Methodologically, we extend the reduced-form model for non-callable bonds to callable bonds by modeling the firm’s call policy, from the market’s perspective, via a stochastic intensity process.\footnote{We recently discovered that Guntay (2002) uses a similar approach to that employed herein. Our reduced-form approach is analogous to the previous use of intensity processes to model both mortgage prepayment risk in valuing residential mortgages and the optimal exercise policy of loan commitments as in Chava and Jarrow (2008).} The existence of a call intensity can be justified on two grounds. First, analogous to the justification of reduced-form credit risk models given by Duffie and Lando (2001) and Cetin, Jarrow, Protter and Yildirim (2004), since the firm’s true value is not observable, the optimal call policy may indeed be viewed as an inaccessible stopping time from the market’s perspective.\footnote{See Li, Liu and Wu (2003) for such a model.} Second, market frictions such as transaction costs, illiquidity and other idiosyncratic reasons may cause premature or delayed call exercise.\footnote{See empirical evidence documented by Vu (1986) and Longsta (1992).} As such, analogous to mortgage bond prepayments, transaction costs can also result in a call-arrival intensity process.\footnote{See Stanton (1995) in this regard.} The existence of a call intensity process allows a reduced-form representation for callable bond valuation.

However, our model is not just a simple extension of the reduced-form model for non-callable bonds, because of important differences between the call and default decisions. When a firm defaults on one of its bonds, due to bond covenants, it defaults on all its bonds. But when a firm decides to call a bond, standard economic theory predicts that it may not call all outstanding bonds. Only bonds whose market value exceeds the call price should be exercised. Consequently, high coupon bonds are more likely to be called than low coupon bonds. Of course, all callable bonds are more likely to be called when interest rates are low. Moreover, at least in theory, there should be a nonlinear relation between call spreads and interest rates due to the nonlinearity embedded in the call option’s payoff. We introduce a parsimonious model for the call spread that: (i) explicitly captures the call spread’s dependence on the coupon rate and its nonlinear dependence on interest rates; and (ii) provides a closed-form approximation for callable bond prices based on a new method developed by Kimmel (2008). In our model, callable corporate bonds...
bond yields impound the joint effects of default and the embedded call provision.

Our new reduced-form model greatly simplifies callable bond pricing and has advantages over existing methods. Unlike the structural approach, our approach circumvents the need to determine the firm’s optimal call policy and thus does not require information on firm value nor capital structure. Unlike Duffie and Singleton’s (1999) American valuation procedure, our approach allows more complex call exercise policies. These more complex exercise policies are consistent with those observed in practice. Furthermore, our approach is less computationally intensive than that of the structural models or Duffie and Singleton’s (1999) procedure. Indeed, in our reduced-form model, the valuation of callable bonds is computationally equivalent to that of Treasury or non-callable bonds, and closed-form solutions are available for popular model specifications. The flexibility and tractability of our reduced-form approach makes it an appealing alternative to existing methods for valuing callable corporate bonds.

We provide one of the first comprehensive empirical analyses of callable bonds under both our reduced-form model and Duffie and Singleton’s (1999) model. Compared to existing studies, our paper considers a relatively large sample of firms that issue both non-callable and callable bonds. We adopt a four-factor affine specification for the default-free term structure, default and call processes. Our results show that the reduced-form model provides a close fit to callable bond prices and has generally smaller in- and out-of-sample pricing errors than does the American option approach of Duffie and Singleton (1999).

While our analysis focuses on the pricing of callable bonds, our reduced-form model has broader applications. For example, our reduced-form model can be used to obtain the risk-neutral call intensities from the market prices of its callable bonds. And based on the estimated call process, one can examine both systematic and firm-specific factors that affect a firm’s ex ante call decision, which have been the focus of most existing studies. The rapid development of the credit derivatives market is providing new data that makes it even more convenient to estimate the default process, providing new data sources for estimating the key parameters in our model.

Our model also simplifies the computation of hedge ratios for callable bonds. The multifactor affine model decomposes callable bond yields into a default-free interest rate, a credit and a call spread. Given the estimated dynamics for each component, it is easy to hedge the risk factor exposures of a callable bond. Indeed, the interest rate and credit risks can be hedged using Treasury and non-callable or credit derivatives, respectively. The reasonably good out-of-sample performance of our model suggests that it is possible to partially hedge call risk using another callable bond from the same firm. In contrast, hedging callable bonds in the model of Duffie and Singleton (1999) crucially depends on a correct modeling of the firm’s optimal call policy, and the calculation of hedge ratios is computationally difficult.
Our reduced-form model can be extended to price options on callable bonds, just as the reduced-form credit risk model facilitates the pricing of credit derivatives. One interesting example is the pricing of options on mortgage-backed securities (MBS), which have become increasingly popular in the fixed-income market (see Zhou and Subramanian (2004)). Mortgage options differ from standard bond options in that their underlying assets have embedded a short call option - prepayment. Similar to that of callable bonds, the prepayment behaviors of MBS are very complicated and depend on factors that are often outside standard economic models. Given our model’s close fit to callable bond prices and the tractability of the affine dynamics, pricing mortgage options using our modeling framework is a natural application. The decomposition of a callable bond’s yield into its different components under our model also makes it much easier to hedge a mortgage option’s exposure to these risk factors.\footnote{Pricing mortgage options based on a given model for the underlying MBS prices (such as in the Black model) does not decompose changes in MBS prices due to different risk factors and thus would not provide hedging strategies as in our model.} In contrast, given the difficulties of the existing models in pricing callable bonds and their computational challenges, it seems unlikely that they can be used to price options on callable bonds.

The remainder of the paper is organized as follows. In Section 2, we develop a reduced-form model for callable corporate bonds. In Section 3, we introduce the data underlying our empirical analysis and discuss the econometric methods for estimation. Section 4 contains the empirical results and Section 5 concludes the paper. In Section 6, the Appendix, we provide a brief discussion on how to apply Kimmel’s (2008) method to our model.

## 2 A Reduced-Form Model for Callable Corporate Bonds

In this section, we develop a reduced-form model for callable corporate bonds by characterizing the call probability as an intensity process. We compare our approach with that of Duffie and Singleton (1999) who price callable bonds as American options. We also provide a closed-form formula for callable corporate bonds under an affine model specification.

### 2.1 The Theory

Let the economic uncertainty be characterized by a filtered probability space \( \{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P\} \) satisfying the usual conditions where \( P \) is the statistical probability measure. We use a triplet \( \{(X, T), (X_c, \tau_c), (X_d, \tau_d)\} \) to denote the cash flows from a zero-coupon callable corporate bond. The first claim \((X, T)\) represents the obligation of the firm to pay \( X \) dollars (=1) at maturity \( T \). The second claim \((X_c, \tau_c)\) defines the stopping time \( \tau_c \) at which the firm calls the bond. In
general, the firm calls the bond when its price is higher than the strike price. When the bond is called, the investors receive the strike price $X_c$, which is represented as a fraction ($k$) of the market value of the bond. We refer to $k$ as the strike-to-market ratio. The third claim ($X_d, \tau_d$) defines the default time $\tau_d$ at which investors receive a residual value $X_d$, which is also represented as a fraction $\delta$ of the market value of the bond. We refer to $\delta$ as the recovery rate. Both the strike-to-market ratio and the recovery rate can be stochastic and depend on market conditions.

Given this notation, the payoff on the zero-coupon callable bond equals

$$ Z = X_c 1_{(\tau_c < \tau_d, \tau_c < T)} + X_d 1_{(\tau_d < \tau_c, \tau_d < T)} + X 1_{(T < \tau_c, T < \tau_d)}; $$

(1)

where $1_{\{\cdot\}}$ is an indicator function. Let $r_t$ be the instantaneous risk-free interest rate. Assuming that both the markets for default-free and corporate bonds are arbitrage-free, the time-$t$ price of this zero-coupon callable corporate bond can be written as

$$ V(t, T, 0, \delta, k) = \begin{cases} \mathbb{E}_t^Q \left\{ Z e^{-\int_t^\tau r_u du} \right\}, & t < \hat{\tau} \\ 0, & t \geq \hat{\tau} \end{cases} $$

(2)

where $\hat{\tau} = \min\{\tau_c, \tau_d, T\}$, and the expectation is taken under the equivalent (to $P$) martingale measure $Q$. We do not assume that these bond markets are complete.\textsuperscript{10}

For valuation purpose, we consider the dynamics of both the call and default intensities under the risk-neutral probability measure $Q$.\textsuperscript{11} Following Lando (1998), we assume that both the call and default point processes, $N_{c,t} \equiv 1_{(t \geq \tau_c)}$ and $N_{d,t} \equiv 1_{(t \geq \tau_d)}$, follow Cox processes. We further assume that the state of the economy is completely characterized by a finite vector of Markov state variables $U_t$ and that the intensities of the call and default processes, $\lambda_{c,t} = \lambda_{c,t}(U_t)$ and $\lambda_{d,t} = \lambda_{d,t}(U_t)$, are functions of the state variable vector. Conditional on the state variables, $N_{c,t}$ and $N_{d,t}$, are independent Poisson processes. As shown later, the state vector in the default process includes two interest rate factors and a firm-specific default factor whereas in the call intensity process it includes a coupon-rate factor and a firm-specific call factor in addition to the two interest rate factors. Intuitively, the probability of a call or default within the next small time interval $\Delta t$ roughly equals $\lambda_{c,t}\Delta t$ or $\lambda_{d,t}\Delta t$, respectively.

By definition, the call and default payoffs $X_c$ and $X_d$, are given by

$$ X_c = k V(\tau_c^-, T, 0, \delta, k) $$

$$ X_d = \delta V(\tau_d^-, T, 0, \delta, k) $$

\textsuperscript{10}The martingale measure employed in the valuation formula is determined by equilibrium in the relevant markets and identified via our estimation procedure below.

\textsuperscript{11}In our empirical implementation, we estimate the model using extended Kalman filter with quasi maximum likelihood. Hence, we also consider the dynamics of the call and default intensities under the physical measure $P$ by assuming constant market prices of risks for term structure, default, and call risks.
where

\[ V (t-, T, 0, \delta, k) = \lim_{s \to t} V (s, T, 0, \delta, k). \]

For the discounted gain process

\[ G_t = e^{-\int_0^t r_u du} V (t, T, 0, \delta, k) (1 - N_{c,t}) (1 - N_{d,t}) + \int_0^t e^{-\int_0^u r_v du} k V (u-, T, 0, \delta, k) (1 - N_{d,t}) dN_{c,u} \]

\[ + \int_0^t e^{-\int_0^u r_v du} \delta V (u-, T, 0, \delta, k) (1 - N_{c,t}) dN_{d,u} \]

to be a martingale, we must have

\[ V (t, T, 0, \delta, k) = E^Q_t \exp \left\{ - \int_t^T (r_u + (1 - \delta) \lambda_{c,u} + (1 - \delta) \lambda_{d,u}) du \right\}. \] (3)

The above formula for callable bond pricing closely resembles that for non-callable bonds contained in Duffie and Singleton (1999). Duffie and Singleton (1999) show that when default arrives with an intensity process, the defaultable bond can be priced as a default-free bond, but with a default-adjusted discount rate: \( r_t + (1 - \delta) \lambda_{d,t} \). The term \((1 - \delta)\) measures the bondholders’ loss when default happens and \(\lambda_{d,t}\) measures the instantaneous probability of default. By discounting future cash flows at a higher discount rate, the reduced-form model of Duffie and Singleton (1999) greatly simplifies credit risk pricing.

Our model extends Duffie and Singleton (1999) to show that the same idea can be applied to bonds with both default risk and an embedded call option. In expression (3), the term \((1 - k)\) measures the bondholders’ loss when the bond is called, and \(\lambda_{c,t}\) measures the call probability. To price callable bonds, we simply used the call- and default-adjusted discount rate: \( r_t + (1 - \delta) \lambda_{c,t} + (1 - \delta) \lambda_{d,t}. \) Similar to Duffie and Singleton (1999), it is also easy to incorporate other market frictions that could affect corporate bond yields, such as illiquidity. And, with appropriate specifications for the dynamics of the spot interest rate, default and call processes (such as affine diffusions), we can easily obtain a closed-form pricing formula for callable bonds.

As shown by Duffie and Singleton (1999), given the price of a zero-coupon callable bond, it is straightforward to value a callable coupon bond. Denote by \( V_c (t, T, c, \delta, k) \) the price of an callable coupon bond with coupon payments of \( c \) on \( T_1, T_2, \ldots, T_n = T \) and a face value of 1. For simplicity of notation, let \( R_u \equiv r_u + (1 - \delta) \lambda_{c,u} + (1 - \delta) \lambda_{d,u}. \) Then, the price of the callable coupon bond equals

\[ V_c (t, T, c, \delta, k) = E^Q_t \left[ \sum_{t < T_i \leq T} ce^{-\int_t^{T_i} R_u du} + e^{-\int_t^T R_u du} \right] \]

\[ = \sum_{t < T_i \leq T} cV (t, T_i, 0, \delta, k) + V (t, T, 0, \delta, k). \] (4)

\(^{12}\)See Duffie and Singleton (1999, footnote 10) for a similar specification.
A callable coupon bond is seen to be a portfolio of zero-coupon callable bonds, all of whose values are given by expression (3) above. This completes our valuation methodology.

Instead of modeling the call probability via an intensity process, Duffie and Singleton (1999) model the call provision as an American option on an otherwise identical non-callable bond where the firm exercises the call provision to minimize the market value of the bond. Denote $R_u^* = r_u + (1 - \delta) \lambda_{d,u}$. Then the price of the (otherwise identical) non-callable bond equals

$$E_t^Q \left[ \sum_{t < T_i \leq T} ce^{-\int_{t}^{T_i} R_u^* du} + e^{-\int_{t}^{T} R_u^* du} \right].$$

(5)

Let $T^* (t, T)$ represent the set of feasible call dates after $t$. Then, the price of the callable corporate bond in Duffie and Singleton’s framework equals

$$V^*_c (t, T, c, \delta, k) = \min_{\tilde{T} \in T^* (t, T)} E_t^Q \left[ \sum_{t < T_i \leq \tilde{T}} ce^{-\int_{t}^{T_i} R_u^* du} + e^{-\int_{t}^{\tilde{T}} R_u^* du} \right],$$

(6)

where we use “*” to differentiate the prices of the callable bond under the two different approaches.

Our reduced-form model has several advantages over that of Duffie and Singleton (1999). First, our pricing equation is easier to implement. For example, if we assume that $R_u$ follows an affine model specification, a closed-form expression for a callable bond’s price is obtainable. As shown later, even for some non-affine specifications of the call spread $(1 - k) \lambda_{c,u}$, we can still obtain a closed-form approximation to a callable bond’s price. In contrast, Duffie and Singleton’s (1999) method requires a numerical computation due to the American feature, even for affine specifications of $R_u^*$. Second, our model is consistent with “suboptimal” exercising strategies for the callable bond. Duffie and Singleton’s (1999) model is not. Third, Duffie and Singleton (1999) make the simplifying assumption that the firm calls the bond to minimize its market value. As pointed out by Duffie and Singleton (1999), “In order to minimize the market value of a portfolio of corporate securities, it may not be optimal for the issuer of the liabilities to call in a particular bond so as to minimize the market value of the particular bond.” Our model does not impose this assumption. Last, Duffie and Singleton (1999) also ignore market frictions, which can significantly alter the optimal call decision. Our method implicitly incorporates any such market frictions within the call intensity. In conjunction, these advantages give our model the potential to fit market data better than that of Duffie and Singleton (1999).

Our model also provides a simpler procedure for computing hedge ratios. Under Duffie and Singleton’s (1999) model, the callable bond contains an embedded American option on an otherwise identical non-callable bond. Thus, the hedge ratio crucially depends on the correct specification of the firms’ optimal call policy, which is difficult to compute. In contrast, in our model, given an appropriate parametric specification of the risk-free interest rate, default and call processes, it is easy to compute a callable bond’s exposure to each risk factor. Then, the risk-free factors can be
hedged using Treasuries, while the credit risk factor can be hedged using a non-callable bond or
credit derivatives. The out-of-sample performance of our model also suggests that it is possible
to partially hedge call risk using another callable bond from the same firm.

Note that even though our empirical analysis focuses on the pricing of callable bonds, our
reduced-form model has broader application. Our model provides a tool for studying corporate
call decisions using the market prices of corporate debt. Given asymmetric information and market
frictions, characterizing a firm’s optimal call decision from basic economic principles is complex.
Consequently, most existing empirical studies have focused on a firm’s ex post call decision. In
contrast, our model makes it possible to infer a firm’s call decision from market prices. That
is, given our assumption on the market price of call risk, we can back out the call intensity
under the physical measure. And, based on the estimated call process, one could examine both
systematic and firm-specific factors that affect a firm’s ex ante call decision. Such investigations
await subsequent research.

Finally, our reduced-form model is useful for pricing options on callable bonds. This is anal-
ogous to the reduced-form credit risk model, which are used to price credit derivatives. Another
application is for pricing mortgage options which are written on TBA passthroughs. The complica-
tion in pricing mortgage options is that their underlying assets are short a prepayment option.
Exercising decisions on the MBS prepayment option are very complicated. The flexibility of our
intensity approach for pricing callable bonds applies to MBS prepayment as well. With affine
specifications of $R_u$, pricing mortgage options is straightforward due to Duffie, Pan, and Sin-
gleton (2000). The decomposition of an MBS yield into in component risks will also facilitate the
hedging of mortgage options.

Before proceeding to model specification, we need to discuss a potential shortcoming of our
approach. In reality, when a bond is called, bondholders receive a contractually specified call
price. In contrast, analogous to the reduced-form literature, our approach assumes that the
bondholders receive a fixed fraction $k$ of the market price of the bond just before being called. This
approximation is imposed to simplify valuation, although it may overestimate callable bond prices
in an environment with unexpectedly low interest rates. In such environments, the likelihood of
calling is high. Therefore, the pre-call bond price is likely to be high as well. To overcome this
difficulty, one could allow $k$ to be stochastic and dependent on the spot rate. This more complex
formulation, however, is left for subsequent research.

2.2 Model Specification

To implement our model, we need to specify the dynamics for the interest rate, default and call
process. We implement a four-factor affine model that captures the joint dynamics of the default-
free term structure, credit spread, and the idiosyncratic component of the call spread. We also
introduce a parsimonious representation for the systematic component of the call spread that incorporates essential elements of the nonlinearity present in call option payoffs. Thus, our model for callable bonds is not just a simple extension of Duffee (1999) for non-callable bonds. Rather, our structure captures some essential differences between call and default decisions.

Following Duffee (1999), Duffie and Singleton (1997), and Duffie, Pedersen, and Singleton (2003), we use a two-factor affine model for the spot interest rate $r_t$:

$$
r_t = \alpha_r + s_{1,t} + s_{2,t},
$$

where $\alpha_r$ is a constant, and the two state variables $\{s_{1,t}, s_{2,t}\}$ represent the slope and level of the Treasury yield curve, respectively. We assume that the dynamics for each of the two factors follow a square root process

$$
ds_i = \kappa_i (\theta_i - s_{i,t}) dt + \sigma_i \sqrt{s_{i,t}} dW_{i,t}, \quad \text{for } i = 1, 2
$$

where $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions under $P$. Under the equivalent martingale measure $Q$, these processes can be represented as

$$
ds_i = (\kappa_i \theta_i - (\kappa_i + \eta_i) s_{i,t}) dt + \sigma_i \sqrt{s_{i,t}} d\hat{W}_{i,t}, \quad \text{for } i = 1, 2
$$

where $\hat{W}_{1,t}$ and $\hat{W}_{2,t}$ are independent standard Brownian motions under $Q$ and $\eta_i$ is the market price of risk associated with $\hat{W}_{i,t}$.

Given enough data, we could separately estimate the recovery rate (strike-to-market ratio) and default (call) intensity. Due to our limited data on the recovery rate and strike-to-market ratio for U.S. corporate bonds, we choose to model $(1 - \delta) \lambda_{d,u} ((1 - k) \lambda_{c,u})$ jointly.

Following Duffee (1999), we assume that default spread has the following representation,

$$
(1 - \delta) \lambda_{d,t} = \alpha_d + h_{d,t} + \beta_d (s_{1,t} - \bar{s}_1) + \beta_d (s_{2,t} - \bar{s}_2),
$$

and

$$
dh_{d,t} = \kappa_d (\theta_d - h_{d,t}) dt + \sigma_d \sqrt{h_{d,t}} dW_{d,t},
$$

where $W_{d,t}$ is a standard Brownian motion under $P$ that is independent of $W_{1,t}$ and $W_{2,t}$. Under the equivalent martingale measure $Q$,

$$
dh_{d,t} = (\kappa_d \theta_d - (\kappa_d + \eta_d) h_{d,t}) dt + \sigma_d \sqrt{h_{d,t}} d\hat{W}_{d,t},
$$

where $\hat{W}_{d,t}$ is a standard Brownian motion under $Q$ that is independent of $\hat{W}_{1,t}$ and $\hat{W}_{2,t}$, and $\eta_d$ is the market price of risk associated with $\hat{W}_{d,t}$.

To simplify the interpretation of the other terms in expression (10), similar to Duffee (1999), we set $\bar{s}_1$ and $\bar{s}_2$ equal to the sample mean of the filtered values of the two state variables $\hat{s}_{1,t}$ and
\( \delta_{2,t} \), respectively. As given, the credit spread contains a constant term \( \alpha_d \), a stochastic term \( h_{d,t} \), and two interest-rate sensitivity components \( \beta_{d1} (s_{1,t} - \bar{s}_1) + \beta_{d2} (s_{2,t} - \bar{s}_2) \). The inclusion of the intercept term \( \alpha_d \) permits a nonzero spread even for firms with close-to-zero default risk. Such a nonzero spread may reflect features such as liquidity or incomplete accounting information. The two interest rate sensitivity components capture the dependence of corporate bond yields on the default-free term structure factors.

Call decisions differ from default decisions in several important ways. First, when a firm defaults on a bond, due to the usual bond covenants, all of its bonds default. But, when a firm calls a bond, it may not call all remaining bonds. Only bonds whose market value exceeds the call price should be exercised. Consequently, high coupon bonds are more likely to be called than low coupon bonds. Of course, all callable bonds are more likely to be called when interest rates are low. Second, unlike the default intensity, because of the nonlinearity present in the call option’s payoffs, there should be a nonlinear relation between the call’s intensity and interest rates.

To capture these differences, we consider the following representation for the call spread. Analogous to the default spread, we assume that the call spread contains both a systematic and an idiosyncratic component. That is,

\[
(1 - k) \lambda_{c,t} = \alpha_c + h_{c,t} + \phi (c, s_{1,t}, s_{2,t}),
\]

where the systematic component \( \phi (c, s_{1,t}, s_{2,t}) \) captures the portion of the call spread due to the standard rational reasons for calling the bond, and the idiosyncratic component \( h_{c,t} \) captures the non-standard considerations. We also include an intercept term \( \alpha_c \) to permit a nonzero call spread even for firms with close-to-zero call risk.

We choose a parsimonious representation for the systematic component \( \phi (c, s_{1,t}, s_{2,t}) \) that allows: (i) explicit dependence of the call spread on the coupon rate, (ii) nonlinear dependence of the call spread on interest rates, and (iii) a closed-form approximation for callable bond prices. Specifically, we assume that

\[
\phi (c, s_{1,t}, s_{2,t}) = \beta_{c1} (s_{1,t} - \bar{s}_1) + \beta_{c2} \frac{c}{s_{2,t}},
\]

where \( c \) is the coupon rate of the callable bond and \( s_{2,t} \) \( (s_{1,t}) \) is the level (slope) factor in our two-factor term structure model.\(^{13}\) The functional form \( \frac{c}{s_{2,t}} \) has been previously used by Richard and Roll (1989) and Gabaix, Krishnamurthy, and Vigneron (2007) to model the nonlinearity of prepayment options in mortgage-backed securities. The functional form \( \frac{c}{s_{2,t}} \) implies that the call premium is high when \( c \) is high and/or \( s_{2,t} \) is low. This captures the idea that all else equal, a high coupon bond is more likely to be called than a low coupon bond, and all callable bonds are

\(^{13}\)We include \( \beta_{c1} \) mainly for completeness. In our empirical analysis, we find that \( \beta_{c1} \) is essentially zero, so our discussion will only focus on the second term.
the equivalent martingale measure \( Q \).

Given the ane structure of the state variables, standard analysis leads to the following solution:

\[
dh_{c,t} = \kappa_c (\theta_c - h_{c,t}) \, dt + \sigma_c \sqrt{h_{c,t}} \, dW_{c,t},
\]

where \( W_{c,t} \) is a standard Brownian motion under \( P \) (independent of \( W_{d,t}, W_{1,t} \) and \( W_{2,t} \)). Under the equivalent martingale measure \( Q \),

\[
dh_{c,t} = (\kappa_c \theta_c - (\kappa_c + \eta_c) \, h_{c,t}) \, dt + \sigma_c \sqrt{h_{c,t}} \, d\hat{W}_{c,t},
\]

where \( \hat{W}_{c,t} \) is a standard Brownian motion under \( Q \) that is independent of \( \hat{W}_{d,t}, \hat{W}_{1,t} \) and \( \hat{W}_{2,t} \), and \( \eta_c \) is the market price of risk associated with \( \hat{W}_{c,t} \).

Furthermore, we assume that the idiosyncratic component of the call spread follows an independent square-root process,

\[
dh_{c,t} = \kappa_c \left( \theta_c - h_{c,t} \right) \, dt + \sigma_c \sqrt{h_{c,t}} \, dW_{c,t},
\]

where \( W_{c,t} \) is a standard Brownian motion under \( P \) (independent of \( W_{d,t}, W_{1,t} \) and \( W_{2,t} \)). Under the equivalent martingale measure \( Q \),

\[
dh_{c,t} = (\kappa_c \theta_c - (\kappa_c + \eta_c) \, h_{c,t}) \, dt + \sigma_c \sqrt{h_{c,t}} \, d\hat{W}_{c,t},
\]

where \( \hat{W}_{c,t} \) is a standard Brownian motion under \( Q \) that is independent of \( \hat{W}_{d,t}, \hat{W}_{1,t} \) and \( \hat{W}_{2,t} \), and \( \eta_c \) is the market price of risk associated with \( \hat{W}_{c,t} \).

Given the above model specification, the default- and call-adjusted discount rate equals

\[
R_u = r_u + (1 - \delta) \lambda_d, u + (1 - k) \lambda_c, u = \alpha_r + s_{1,u} + s_{2,u} + \left[ \alpha_d + h_{d,u} + \beta_1 (s_{1,u} - \bar{s}_1) + \beta_2 (s_{2,u} - \bar{s}_2) \right] + \left[ \alpha_c + h_{c,u} + \beta_1 (s_{1,u} - \bar{s}_1) + \beta_2 \frac{c}{s_{2,u}} \right] = A + s_{1,u}^* + (1 + \beta_2) s_{2,u} + \beta_2 \frac{c}{s_{2,u}} + h_{d,u} + h_{c,u},
\]

where \( A = \alpha_r + \alpha_d + \alpha_c - \beta_1 \bar{s}_1 - \beta_2 \bar{s}_1 - \beta_2 \bar{s}_2 \) and \( s_{1,u}^* = [1 + \beta_1] s_{1,u} \). The dynamics of the translated factor \( s_{1,t}^* \) under the measures \( P \) and \( Q \) are, respectively

\[
ds_{1,t}^* = \kappa_1 \left( \theta_1^* - s_{1,t}^* \right) \, dt + \sigma_1^* \sqrt{s_{1,t}^*} \, dW_{1,t},
\]

\[
ds_{2,t}^* = \left( \kappa_1 \theta_1^* - (\kappa_1 + \eta_1) s_{1,t}^* \right) \, dt + \sigma_1^* \sqrt{s_{1,t}^*} \, dW_{1,t},
\]

where \( \theta_1^* = \theta_1 (1 + \beta_1 + \beta_c^1) \) and \( \sigma_1^* = \sigma_1 \sqrt{1 + \beta_1 + \beta_c^1} \).

Thus, we have

\[
V(t, T, 0, \delta, k) = E_t^Q \left[ \exp \left( - \int_t^T R_u \, du \right) \right] = \exp \left[ - A (T - t) \right] \cdot E_t^Q \left[ \exp \left( - \int_t^T s_{1,u}^* \, du \right) \right]
\]

\[
\cdot E_t^Q \left[ \exp \left( - \int_t^T h_{d,u} \, du \right) \right] \cdot E_t^Q \left[ \exp \left( - \int_t^T h_{c,u} \, du \right) \right] \cdot \pi(s_{2,t}, t, T)
\]

where

\[
\pi(s_{2,t}, c, t, T) = E_t^Q \left[ \exp \left( - \int_t^T \left( (1 + \beta_2) s_{2,u} + \beta_2 \frac{c}{s_{2,u}} \right) \, du \right) \right].
\]

Given the affine structure of the state variables, standard analysis leads to the following solution:

\[
V(t, T, 0, \delta, k) = \exp \left\{ \psi_0(t) - \psi_1(t) s_{1,u}^* - \psi_d(t) h_{d,u} - \psi_c(t) h_{c,u} \right\} \cdot \pi(s_{2,t}, c, t, T),
\]

(17)
where \( \psi_0 (t) = \psi_{01} (t) + \psi_{0c} (t) + \psi_{0d} (t) \),

\[
\begin{align*}
\psi_1 (t) &= \frac{2 \left( e^{\gamma_1 (T-t)} - 1 \right)}{2\gamma_1 + (\kappa_1 + \eta_1 + \gamma_1) \left( e^{\gamma_1 (T-t)} - 1 \right)}, \\
\psi_{01} (t) &= \frac{2\kappa_1 \theta_1^*}{\sigma_1^2} \log \left[ \frac{2\gamma_1 e^{\frac{1}{2} (\kappa_1 + \eta_1 + \gamma_1)}}{2\gamma_1 + (\kappa_1 + \eta_1 + \gamma_1) \left( e^{\gamma_1 (T-t)} - 1 \right)} \right],
\end{align*}
\]

and for \( j = c \) and \( d \),

\[
\begin{align*}
\psi_j (t) &= \frac{2 \left( e^{\gamma_j (T-t)} - 1 \right)}{2\gamma_j + (\kappa_j + \eta_j + \gamma_j) \left( e^{\gamma_j (T-t)} - 1 \right)}, \\
\psi_{0j} (t) &= \frac{2\kappa_j \theta_j}{\sigma_j^2} \log \left[ \frac{2\gamma_j e^{\frac{1}{2} (\kappa_j + \eta_j + \gamma_j)}}{2\gamma_j + (\kappa_j + \eta_j + \gamma_j) \left( e^{\gamma_j (T-t)} - 1 \right)} \right],
\end{align*}
\]

\( \gamma_i = \sqrt{(\kappa_i + \eta_i)^2 + 2\sigma_i^2} \) and \( \gamma_j = \sqrt{(\kappa_j + \eta_j)^2 + 2\sigma_j^2} \).

It is difficult to obtain a closed-form solution for \( \pi (s_{2,t}, c, t, T) \) because the discount factor \( (1 + \beta_{d_2}) s_{2,u} + \beta_c c_{s_{2,u}} \) is outside the standard affine family. Based on a new method developed by Kimmel (2008), we obtain a closed-form power series expansion of \( \pi (s_{2,t}, c, t, T) \), which leads to a closed-form approximation to the price of the zero-coupon callable bond, \( V (t, T, 0, \delta, k) \). For more detailed discussion of this method, see the appendix.

Based on this pricing formula for the zero-coupon callable bond, we obtain the price of a regular callable bond using expression (4). One complication is that if we include \( \beta_{c2} \) to discount both the coupons and the principal, then the callable bond price converges to zero when \( s_{2,t} \to 0 \), because in such case \( \frac{c}{s_{2,t}} \to \infty \). This fact suggests that \( \frac{c}{s_{2,t}} \) will probably discount future cash flows too heavily (due to the call spread) when \( s_{2,t} \) is low.\(^\text{14}\)

To overcome this difficulty, we chose to price the coupons using the formula in expression (17) and to price the principal as if it is a zero-coupon non-callable bond (this is equivalent to setting \( k = 1 \) in \( R_u \)). We believe that this approach provides a reasonable approximation to callable bond pricing. Indeed, there are two different, but yet equivalent ways of modeling the cash flows bondholders receive when a bond is called. The standard approach is that when a bond is called, bondholders give up both the future coupons and the principal for the call price. The standard approach implies that we should discount both the coupons and the principal using the same call spread. An alternative approach for modeling the bond’s call provision is to assume that when a bond is called, the bondholders: (i) lose all the future coupons; but (ii) exchange the principal for the call price. This alternative approach implies that we should discount the coupons and the

\(^{14}\) One possible solution is to assume that the systematic component of the call spread has the functional form of \( \frac{c}{s_{2,t} + d} \), for \( d > 0 \). This naturally reduces the discounting when \( s_{2,t} \) is low because the call spread converges to a constant when \( s_{2,t} \to 0 \). Unfortunately, Kimmel’s (2008) method does not apply to the case where \( d > 0 \).
principal using different call spreads. Coupons are discounted more heavily because they are lost; while the principal is discounted less heavily because it is exchanged for the call price. We employ this alternative approach.

One potential shortcoming of this alternative approach is that it might underprice a callable bond because the strike-to-market ratio $k$ is likely to be bigger than 1, given that the bondholders give up the principal (which is due at maturity) for the call price (which is received when a bond is called). However, this underpricing is likely to be approximately constant due to two offsetting factors. One, when interest rates are low, a callable bond is more likely to be exercised and the time difference between the (stochastic) call date and the maturity date is likely to be bigger. Second, lower interest rates also means that the difference in terms of present values becomes smaller. Combined, these two observations imply that this potential underpricing should be small and easily captured by the constant term in the call premium (i.e., $\alpha_c$).

3 The Data and Estimation Method

3.1 Research Design

In this section, we discuss the data and the econometric method for implementing our model and that of Duffie and Singleton (1999) for callable bonds. Callable bond yields have three components: the risk-free interest rate, the credit and call spreads. We use a two-stage estimation procedure in the determination of these three components. First, we estimate the default-free term structure using yields on Treasury securities. Then, we use these estimated risk-free processes to price both non-callable (credit risk) and callable bonds (call risk). As pointed out by Duffee (1999), due to the huge dimension of the problem, it is computationally infeasible to jointly estimate the default-free term structure in conjunction with the estimation of the corporate bond parameters using all of the firms in our sample. The alternative of estimating the default-free term structure jointly with only the corporate bonds from each individual firm is also unattractive because it generates different risk-free process estimates for different firms.

To simplify the implementation, we first estimate the default process using a non-callable bond with similar maturity.\footnote{We could estimate the defaultable term structure using non-callable bonds with different maturities from the same firm. However, part of the credit and call spreads could be due to liquidity risk and bonds with different maturities may have different liquidity. By using a non-callable bond whose maturity is similar to that of the callable bond, we hope to minimize the differences in liquidity between the two bonds.} In our reduced-form model, this allows us to separate the default and call risks for the callable bond. Otherwise, by using the callable bond alone, due to the common factors that affect both risk-neutral default and call intensities, it would be difficult to separately estimate default and call spreads. This procedure also facilitates the comparison of
our method with that of Duffie and Singleton (1999). With the estimated default process of the non-callable bond, we can then compute the price of the callable bond as an American option written on this similar non-callable bond. Estimating parameters using only the callable bond under the model of Duffie and Singleton (1999), though possible, would be a numerical challenge. As a result, Berndt (2004) considers only one firm when implementing this estimation approach for Duffie and Singleton’s (1999) model. Finally, given the estimated default-free and defaultable models, we then estimate the call process using the callable bond prices in our model, and the callable bond prices under Duffie and Singleton’s (1999) model using the least squares method of Longstaff and Schwartz (2001). We refer to Berndt (2004) for a more complete description of this method.

For firms with multiple callable bonds outstanding, we allow each bond to have its own call process. This assumption is similar to that used by Duffie, Pedersen, and Singleton (2003) in their analysis of Russian government bonds. Duffie, Pedersen, and Singleton (2003) argue that due to political and other factors, the Russian government might have different considerations when choosing which bond to default. Similarly, there could be different considerations why a firm decides to call a particular bond. Multiple callable bonds from the same firm also make it possible to study the model’s out-of-sample performance. That is, we can use the estimated call process of a callable bond to price another callable bond from the same firm with similar characteristics. Duffie, Pedersen, and Singleton (2003) provide similar out-of-sample analysis using Russian government bonds.

3.2 The Data

Our empirical analysis requires three types of data: yields on default-free bonds, and yields on non-callable and callable corporate bonds from the same firm with similar maturities. We use the unsmoothed Fama-Bliss spot rates from January 1982 to December 1996 to estimate the default-free term structure.\(^{16}\) The data include spot rates for 6-month and 12-month Treasury bills, and 2-, 3-, 5-, 10-, and 30-year Treasury bonds. The maturities of the Treasury securities cover the maturity spectrum of all corporate bonds used in our analysis. The sample period also covers that of the corporate bonds used in our analysis.

Corporate bond data are obtained from the Lehman Brothers Fixed Income Database distributed by Warga (1998). This database contains monthly price, accrued interest, and return data on corporate bonds up to December 1996. We use only corporate bonds with prices based on dealer quotes.\(^ {17}\) In the Lehman dataset, there are in total 5,009 firms and 32,295 bonds from 1973 to 1996. We find 12,509 callable and 8,888 non-callable bonds with no other provisions. As

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\(^{16}\)We thank Robert Bliss for providing us this data set, see Bliss (1997).

\(^{17}\)Sarig and Warga (1989) have shown that matrix prices are problematic.
mentioned previously, callable bonds represent a significant proportion of traded corporate bonds. To ensure accurate parameter estimates, we require the bonds included in our sample to have at least 36 months of trader-quoted prices.\textsuperscript{18} To implement the three-step estimation procedure, we also require a firm to have at least one non-callable and one callable bond. With this restriction, we have in total 131 firms. To implement the two models, the observations of the non-callable and callable bonds must be in the same time period and the maturities of the two bonds must be similar. So, we further require that there are at least 24 months in which the prices of both the non-callable and callable bonds are observed and the maturity difference between the two bonds is less than 20 months. After these restrictions, we have 34 firms left in our sample, whose names and numbers of non-callable and callable bonds outstanding are given in Table 1.\textsuperscript{19} The restrictions imposed on the data facilitate joint estimation of the call and default process parameters. However, it may induce a potential sample selection bias toward large and high-levered firms with more actively traded bonds.

Panels A and B of Table 2 provide summary statistics of the non-callable and callable bonds for the 34 firms used in our analysis, respectively. In our pricing analysis, we focus on one non-callable and one callable bond from each firm. Thus, the summary statistics in Panels A and B of Table 2 are for 34 non-callable and callable bonds, respectively. Specifically, we focus on the credit ratings, prices, coupons, yield to maturities, and maturities of the bonds. The median credit rating for both the straight and callable bonds is A, and most bonds have ratings above BBB. This is not surprising because the Lehman dataset mainly covers investment grade bonds. We also have some high yield bonds with credit rating as low as BB. Given the strong credit ratings of most bonds, most prices are close to or above par. The coupon rates of most bonds are between 8 and 14%. Yield to maturities of non-callable and callable bonds have a mean of 8.5% and 9.2%, respectively. As shown therein, callable bonds have higher yields to maturities than non-callable bonds. The mean (median) maturities of both non-callable and callable bonds are about 9 years (6 years), while the shortest (longest) bond has a maturity of 1 (30) year(s).

3.3 The Estimation Method

There are different econometric methods that one can use to implement the three-step estimation procedure. Instead of using maximum likelihood which typically assumes that certain yields are observed without errors and then obtain the state variables by inversion, we use the extended Kalman filter which assumes that all yields are observed with measurement errors. The Kalman filter approach has been previously used by many authors, e.g. Duffee (1999).

\textsuperscript{18}A similar requirement was used by Duffee (1999) in his analysis of non-callable bonds where he finds that 161 firms satisfy the requirement.

\textsuperscript{19}Because of our model specification, we focus on only callable coupon bonds in our empirical analysis.
Let \( Y_t = (y_{1t}, ..., y_{Nt})' \) be the yields on \( N \) Treasury bonds, and \( S_t = \{s_{1,t}, s_{2,t}\} \) be the state variables that drive the default-free term structure. Then, we have the following measurement and transition equations:

\[
Y_t = \Phi(S_t) + \varepsilon_t, \quad E_{t-1} (\varepsilon_t \varepsilon_t') = \Sigma
\]

\[
S_t = \mu + \Gamma S_{t-1} + v_t, \quad E_{t-1} (v_t v_t') = \Omega(S_{t-1})
\]

where \( \Phi(\cdot) \) maps the state variables into yields of Treasury bonds, and \( \Sigma \) and \( \Omega(S_{t-1}) \) are diagonal matrices corresponding to the variances of the measurement errors of the yields and state variables, respectively. The expectations in the above two equations are taken under the statistical probability measure. In total, we use monthly yields on seven Treasury bonds with maturities of 6, 12 months, 2, 3, 5, 10, and 30 years in our estimation.

In the transitional equation, \( \mu \) and \( \Gamma \) are defined as,

\[
\mu = \begin{pmatrix}
\theta_1 \left( 1 - e^{-\kappa_1/12} \right) \\
\theta_2 \left( 1 - e^{-\kappa_2/12} \right)
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
e^{-\kappa_1/12} & 0 \\
0 & e^{-\kappa_2/12}
\end{pmatrix},
\]

and \( \Omega(S_{t-1}) \) is a 2 x 2 diagonal matrix with elements

\[
\Omega_{i,i} (S_{t-1}) = \kappa_i^{-1} \sigma_i^2 \left[ s_{i,t-1} \left( e^{-\kappa_i/12} - e^{-2\kappa_i/12} \right) + \frac{\theta_i}{2} \left( 1 - e^{-\kappa_i/12} \right)^2 \right] \text{ for } i = 1, 2.
\]

Given the estimated parameters and the state variables \( \{\hat{s}_{1,t}, \hat{s}_{2,t}\} \) of the interest rate process, we can estimate the default process of a particular firm using the prices of a non-callable bond. Let \( Y_{d,t} \) represent the yield of the defaultable bond. Then we have the following measurement and state equations:

\[
Y_{d,t} = \Phi_d (h_{d,t}, \hat{s}_{1,t}, \hat{s}_{2,t}) + \varepsilon_{d,t}, \quad E_{t-1} (\varepsilon_{d,t} \varepsilon_{d,t}') = \Sigma_d,
\]

\[
h_{d,t} = \mu_d + \Gamma_d h_{d,t-1} + u_t, \quad E_{t-1} (u_t u_t') = \Omega_d (h_{d,t-1}),
\]

where \( \Phi_d \) maps the state variables \( h_{d,t}, \hat{s}_{1,t}, \) and \( \hat{s}_{2,t} \) into the non-callable bond yields.

Based on the estimated default parameters and intensity \( \hat{h}_{d,t} \) from the second step, we estimate the call process using callable bond prices. The measurement and transition equations of the callable bond yields \( Y_{c,t} \) are given by

\[
Y_{c,t} = \Phi_c (h_{c,t}, \hat{h}_{d,t}, \hat{s}_{1,t}, \hat{s}_{2,t}) + \varepsilon_{c,t}, \quad E_{t-1} (\varepsilon_{c,t} \varepsilon_{c,t}') = \Sigma_c,
\]

\[
h_{c,t} = \mu_c + \Gamma_c h_{c,t-1} + \xi_t, \quad E_{t-1} (\xi_t \xi_t') = \Omega_c (h_{c,t-1}).
\]

In these equations, the function \( \Phi_c (h_{c,t}, \hat{h}_{d,t}, \hat{s}_{1,t}, \hat{s}_{2,t}) \) maps the call and default intensities, \( h_{c,t} \) and \( h_{d,t} \), and the two term structure factors \( \hat{s}_{1,t} \) and \( \hat{s}_{2,t} \) into the callable bond yields. Both mappings, \( \Phi_d \) and \( \Phi_c \), are implicitly given by numerically solving for the non-callable and callable
bond yields given the state variables. \( \Sigma_c \) is the measurement error variance for the callable bonds’ yields.

The transition equation components \( \mu_j \) and \( \Gamma_j \) are given by

\[
\mu_j = \theta_j \left( 1 - e^{-\kappa_j} \right), \quad \Gamma_j = e^{-\kappa_j}, \quad \text{for } j = c, d.
\]

The variances \( \Omega_d \) and \( \Omega_c \) associated with the default and call processes are given by

\[
\Omega_j (h_{j,t-1}) = \kappa_j^{-1} \sigma_j^2 \left[ h_{j,t-1} \left( e^{-\kappa_j} - e^{-\kappa_d} \right) + \frac{\theta_j}{2} \left( 1 - e^{-\kappa_j} \right)^2 \right] \quad \text{for } j = c, d.
\]

It is well known that in theory the extended Kalman filter does not provide consistent parameter estimates in our setting. However, many studies, such as Lund (1997), Duan and Simonato (1999), Duffee and Stanton (2004), and Trolle and Schwartz (2008), have investigated the finite sample performances of the Kalman filter and have found very small biases in parameter estimates in various term structure applications. In the appendix, we also show that the extended Kalman filter leads to accurate parameter estimates of the models of default-free, defaultable, and callable bonds in finite samples.

4 The Empirical Results

This section presents the empirical evidence on the pricing performance of our reduced-form model for callable corporate bonds. After discussing the estimation of the default-free and defaultable term structures, we focus on fitting the callable bond prices. We also compare our model with that of Duffie and Singleton (1999) for several firms with enough information for both an in- and out-of-sample analysis. Then, we assess the importance of a liquidity premium in callable bond spreads. Finally, we provide some diagnostics on the pricing errors for the callable bonds.

4.1 Overall Performance

Table 3 reports the parameter estimates for the default-free term structure. We obtain similar results to Duffee (1999), who estimates the same model for a slightly different sample period. The two factors have the natural interpretation as the slope and level factors of the yield curve. We see that the first factor is significantly negatively correlated with the spread between the 30-year and 6-month yields, while the second factor is highly correlated with the 30-year yields. We refer to the two factors as the slope and level factors for the rest of the paper. The parameter estimates of the two factors show that the first factor has a higher long-run mean and a faster speed of mean-reversion. The second factor behaves more like a martingale, with a lower long-run mean and a slower speed of mean-reversion. The estimates of \( \eta_j \)'s are negative for both factors (though not all significant), suggesting negative risk premiums associated with both factors. While the two-factor
model might not be able to fully capture the dynamics of the default-free term structure, similar to Duffee (1999), we find that it provides a reasonably good fit to the data.

Next, based on the estimated default-free process, we estimate the default risk parameters using the non-callable bonds for each of the 34 firms. Panel A of Table 4 reports the median, mean and interquartile range for each estimated parameter, the fitted values of the default spread \((1 - \delta) \lambda_d\), and the square root of the mean squared differences between the actual and fitted bond prices (RMSE). The fitted value is the filtered prediction from the Kalman filter. The parameter estimates are not as noisy as those of Duffee (1999), possibly due to the smaller number of firms considered in our sample. We find that the firm-specific default intensity, \(h_d\), is mean-reverting and commands a negative risk premium. The median estimates of \(\beta_{d1}\) and \(\beta_{d2}\) show that the slope factor has a significantly negative impact on the credit spread, while the effect of the level factor on the credit spread is economically close to zero. The median estimate of \(\beta_{d1}\) implies that a 100 basis points decline in \(s_{1,t}\) increases the credit spread \((1 - \delta) \lambda_d\) by 124.8 basis points. In contrast, the median estimate of \(\beta_{d2}\) implies that a 100 basis points decline in \(s_{2,t}\) decreases the default spread by 6 basis points. The slope of the yield curve is a good indicator of the state of the economy. The short-end of the yield curve declines and the slope steepens in recessions, which coincides with a high default risk in corporate bonds. The median fitted value of the credit spread \((1 - \delta) \lambda_d\) per annum is about 109 basis points. The three-factor affine model fits defaultable bond prices well. Indeed, the median and mean RMSEs as a percentage of the principal are only 1.3 and 2.2 basis points, respectively.

Given the estimated default-free and defaultable term structures from the first two steps, next, we estimate the call process using a callable bond from the same firm with similar maturity. Panel B of Table 4 reports the median, mean, and interquartile range of the estimated parameters, the fitted values of the call spread \((1 - k) \lambda_c\), and the RMSEs of the callable bonds. The estimated mean reversion parameter \((\kappa_c)\) and risk premium \((\eta_c)\) of the call intensity process are smaller than those of the default process. This could be due to the fact that losses from a call are typically smaller than those from a default. Both the mean and median estimates of \(\beta_{c1}\) for the call spread process are almost zero. Thus, while the slope factor has a significant negative impact on the default spread, its impact on the call spread is minimal. In contrast, the level factor has a significant and nonlinear impact on the call spread. The level of the long rate affects a firm’s refinancing costs and therefore directly affects the firm’s call decision. Moreover, the lower the level of the interest rate, the more likely a firm will call its bonds, and thus the higher the call spread.

In our model, \(\beta_{c2} \frac{\eta_c}{s_{2,t}}\) naturally captures this nonlinear relation between the call spread and interest rates. Suppose \(c = 8.63\%\) (the mean coupon rate in Panel B of Table 2), then the median estimate of \(\beta_{c2}\) implies that a 100 basis points decrease in \(s_{2,t}\) from 10% to 9% leads to a 3.5 basis...
points increase in the call spread. However, a 100 basis points decrease in $s_{2,t}$ from 2% to 1% leads to a 160 basis points increase in the call spread. A nonlinear relation between a 100 basis point increase and decrease. The mean fitted value of call spread $(1 - k) \lambda_c$ is about 50 basis points per annum, which is about half of that of the credit spread $(1 - \delta) \lambda_d$. Our reduced-form model also has small pricing errors for callable bonds. Indeed, the median and mean RMSEs as a percentage of the principal are 2.5 and 3.6 basis points, respectively.

The above results show that the reduced-form model for the prices of non-callable bonds has a good in-sample fit. The pricing errors for most bonds are less than 5 basis points. This accurate pricing makes it possible to estimate call spreads precisely. Based on the estimated call spreads, we could examine possible economic factors, both systematic (such as term structure factors considered in our model) and firm-specific factors. Thus, our model makes it possible to utilize the rich information from callable bonds to analyze firm’s \textit{ex ante} call decisions. This extension, however, awaits subsequent research.

4.2 A Comparison with Duffie and Singleton (1999)

In this section, we compare the performance of our reduced-form model with that of Duffie and Singleton (1999) in pricing callable bonds. For robustness, we consider both in- and out-of-sample tests. For a firm with multiple callable bonds, we estimate our reduced-form model using one of the callable bonds to obtain the in-sample results. Then, based on the estimated call process, we price another callable bond from the same firm with similar coupon and maturity to obtain the out-of-sample results. This procedure is similar to that of Duffie, Pedersen, and Singleton (2003) in their out-of-sample analysis of Russian government bonds.

For our analysis, we need at least three bonds, one non-callable and two callable bonds, from the same firm that have similar maturities and observations over the same time period. There are 15 bonds from five firms (DuPont, Marriott, Occidental, Sears Roebuck, and Xerox) in our sample that satisfy this data requirement. We also price the callable bonds using Duffie and Singleton’s (1999) method based on the estimated default-free and defaultable processes.

Panel A of Table 5 reports the in-sample pricing errors based on our model for the callable bonds of the five firms. The coupon rates of these bonds range from 8% to 9.875%, while the maturities range from 5.8 to 10.5 years. Not surprisingly, the reduced-form model has very small pricing errors for the five callable bonds. Pricing errors range from a low of 2 basis points to a high of 6 basis points as a percentage of principal. Interestingly, given the distribution of the pricing errors of the 34 firms, the pricing errors of the five callable bonds have a roughly even distribution among the quartiles. In contrast, the method of Duffie and Singleton (1999) has much larger pricing errors for each of the callable bonds, suggesting that exercise decisions differ significantly from the rational ones assumed in the model.
The first column of Figure 1 presents the time series plots of the market prices, and the prices predicted by the reduced-form and Duffie and Singleton’s (1999) model over the sample period for each of the five firms. The figures confirm our conclusion that the reduced-form model prices callable bonds better than the American option approach. For most firms, the true prices and the predicted prices from the reduced-form model are almost indistinguishable. In contrast, the American option approach has relatively large pricing errors for certain bonds, especially for Sears Roebuck and Xerox, and to a lesser extent, Occidental. This evidence supports the robustness of our model in capturing call decisions that might not be “rational” in the traditional sense.

Figure 2 documents the performance of the two models in capturing the relation between the call spread and interest rates. To capture the nonlinear dependence of the call spread on interest rates, the reduced-form model relies on the functional form \( \beta c \frac{z}{x_1 t} \), while the Duffie and Singleton model explicitly prices the embedded American call option. For each callable bond, we plot the model-implied call spread at each point in time against the difference between the yield of an otherwise identical non-callable bond and the coupon rate of the callable bond. Thus the smaller the difference, the deeper the call option is in the money.

The first (second) column of Figure 2 compares the actual data with the predictions from the reduced-form (and the American option) approach. Except for DuPont, we find a clear positive relation between the call spread and the moneyness of the call option. Indeed, the call spread is higher when the call option is deeper in the money. For bonds that have a relatively wider range of moneyness, namely Marriott and Xerox, we find a nonlinear relation between the call spread and moneyness: Except for a few Marriott outliers, the general pattern is that the call spread is nearly zero when the bond is out of the money and it increases nonlinearly when the bond becomes in the money. Interestingly, both the reduced-form model and the American option approach capture the nonlinear relation between the call spread and moneyness well. Because of the idiosyncratic call component, the reduced-form model is more flexible and thus captures the data better than the American option approach.

Based on the estimated call spread of the first callable bond, we price the second callable bond from the same firm. We call this “out-of-sample” analysis because the second callable bond has not been used in model estimation. We also price the second callable bond using the American option approach based on the estimated default-free and defaultable term structures. Given that the American option approach has relatively large pricing errors for the in-sample bonds, we modify the call price of each bond so that the method has a zero average in-sample pricing error. Then we price the out-of-sample bond based on the modified call price. In some sense, this approach provides a fairer comparison between the reduced-form and the American option approach.

Panel B of Table 5 shows that the out-of-sample pricing errors of the reduced-form model are much larger than their in-sample pricing counterparts. This should not be surprising given that
these two callable bonds are not exactly the same and therefore the idiosyncratic components of their call spreads can be different. However, the pricing errors of the reduced-form model are still generally smaller than that of the American option approach. This is again confirmed by the time series plots of the observed and fitted prices in the second column of Figure 1. In general, the reduced-form model can price the second callable bond better than the American option approach. For example, for DuPont, Marriott, and Xerox, the market and predicted prices from the reduced-form model are quite close to each other. But for Xerox, the American option approach significantly underprices (overprices) the second callable bond during the first (second) half of the sample. Both the reduced-form model and the American option approach have relatively large pricing errors for Occidental and Sears Roebuck. The adjustment of call price does not necessarily reduce the pricing errors for all bonds, because the American option approach does not uniformly over- or under-price the in-sample bonds. Figure 3 shows the out-of-sample performance of both models in capturing the relation between the call spread and interest rates. While both models do not perform as well as in-sample, they capture the nonlinear relation between the call spread and interest rates for Marriott, and in particular Xerox, reasonably well.

In summary, the above analysis confirms that the reduced-form model is reasonably successful in pricing callable bonds in both in-sample and out-of-sample settings. Its flexibility in modeling call and its tractability in pricing applications give it certain advantages over the American option approach. The superior out-of-sample performance also suggests that the reduced-form model could be useful for pricing a new callable bond yet to be issued by a firm with no other callable bonds outstanding. One could first find a matching firm (a firm with similar characteristics as the issuer of the callable bond) with an outstanding callable bond that has similar characteristics as the callable bond to be issued. Then one could use the implied call intensity of the traded callable bond to price the new callable bond to be issued.

4.3 The Importance of a Liquidity Premium

While credit spreads could be due to risk factors other than default, such as liquidity, existing implementations of the reduced-form model using corporate and sovereign bonds, such as Duffee (1999) and Duffie, Pedersen, and Singleton (2003), typically do not separately estimate the liquidity component in credit spreads.\(^20\) One reason could be lack of data. For example, the Lehman Brothers dataset does not contain any information on the liquidity of corporate bonds, such as bid-ask spreads, depth, or trading volume. The ability to incorporate non-default and non-call factors into yield spreads is an advantage of the reduced-form model. In contrast, it is not obvious how to incorporate liquidity risk for callable bonds in the American option approach. However,\(^20\)

\(^20\) There are studies, such as Longstaff, Neis, and Mithal (2005), that try to separate liquidity and credit risks using prices of both credit derivatives and corporate bonds.
this observation also raises the question as whether our estimated call spread is actually due to liquidity risk, or the differentials in liquidity between non-callable and callable bonds. This section provides some analysis on the possible magnitude of the liquidity premium in callable bond spreads.

The yield spread of a callable bond reflects credit, call and liquidity risk. If we allow each bond to have its own process for each component, then due to data limitations and common factors underlying most components, identifying these risks from just bond prices is difficult. Instead, we make some simplifying assumptions and try to separate the call and liquidity premiums using three bonds, one straight and two callable bonds with similar maturities from the same firm.

Specifically, we assume that all three bonds share the same default process, estimated from the straight bond. As the two callable bonds have similar maturities and coupon rates, we assume that their call spreads are proportional to each other. That is, \((1 - k_1) \lambda_{1c} = \rho (1 - k_2) \lambda_{2c}\), where \(\rho\) is slightly bigger or smaller than one. We also assume that the two bonds share the same liquidity spread \(l_t\), which follows a square-root process

\[
dl_t = \kappa_l (\theta_l - l_t) \, dt + \sigma_l \sqrt{l_t} \, dW_{l,t}
\]

where \(W_{l,t}\) is a standard Brownian motion and is independent of all other Brownian motions introduced before. Then the default-, call-, and liquidity-adjusted discount rate of the first callable bond is

\[
R_u = r_u + (1 - \delta) \lambda_{d,u} + (1 - k_1) \lambda_{1c,u} + l_u.
\]

Unlike the call and default spread, we do not include systematic factors in the liquidity process for two reasons. First, we feel that the liquidity differentials between callable and non-callable bonds may be due to specific characteristics of the bonds, and thus are non-systematic. Second, if similar systematic factors affected both the call and liquidity process, then it would be empirically difficult to separate the two.

Due to data requirements, we focus on the same five firms. Table 6 reports the mean of parameter estimates of the liquidity process from the five firms. Note that the long-run mean of the liquidity process is very small, only about 4 basis points. The process is mean-reverting and there is a negative risk premium associated with liquidity risk. Figure 4 provides time series plots of the observed and fitted prices of the two callable bonds from the five firms and the term structure of liquidity spreads. The latter is calculated based on the estimated liquidity parameters and the sample mean of the filtered liquidity variables, \(\hat{l}_t\). Even though we assume a proportional call intensity, our model fits the two callable bond prices reasonably well. The term structure of liquidity spread is seen to be upward sloping. Consistent with the parameter estimates, the magnitude of the liquidity premium is quite small, approximately a few basis points. This amount is economically less significant than that of the default and call premiums, which are about 100
and 50 basis points, respectively. Therefore, the call spreads estimated without adjusting for liquidity risk appear to be a reasonable reflection of call risk.

While the above results are based on restrictive assumptions, there are other reasons to believe that the liquidity differentials between callable and non-callable bonds may not be substantial. In a recent study using the TRACE (Trade Reporting and Compliance Engine) dataset, Edwards, Harris, and Piwowar (2007) show that bonds with attached calls and puts have significantly lower estimated transaction costs than those without such features in 2003. They attribute their results to the fact that with low interest rates in 2003, the callable bonds are more likely to be called. Although their sample is very short, the evidence suggests that callable bonds are not less liquid than non-callable bonds.

4.4 Diagnostics

This section provides further diagnostics on the implied state variables and pricing errors for callable bonds under our reduced-form model to identify potential sources of model misspecification.

First, our model assumes that $h_{ct}$ and $s_{2,t}$ are independent of each other. We check to see whether the correlation coefficient between changes of the two filtered state variables, $\text{Corr} \left( \Delta \hat{h}_c, \Delta \hat{s}_2 \right)$, is zero and exhibits any systematic bias. Figure 5 plots $\text{Corr} \left( \Delta \hat{h}_c, \Delta \hat{s}_2 \right)$ for each callable bond against the average moneyness of the bond during the sample period (measured by the average yield to maturity of an otherwise identical non-callable bond minus the coupon rate of the callable bond). We find some evidence of non-zero correlation between $\Delta \hat{h}_c$ and $\Delta \hat{s}_2$. Specifically, there is a negative relation between $\text{Corr} \left( \Delta \hat{h}_c, \Delta \hat{s}_2 \right)$ and the average moneyness of a callable bond: on average $\text{Corr} \left( \Delta \hat{h}_c, \Delta \hat{s}_2 \right)$ is more negative (positive) for bonds that are deeper in-the-money (out-of-the-money). One possible reason for this result could be that our nonlinear model generates too low (high) systematic call spreads when interest rates are low (high). To compensate for this, the idiosyncratic call spread needs to be higher (lower) when interest rates are low (high), or equivalently when the callable bond is in-the-money (out-of-the-money). This suggests that a nonlinear functional form that goes to $\infty$ (0) faster than $\frac{c}{s_{2,t}}$ when $s_{2,t} \to 0$ ($\infty$) might be able to better capture the data.

Next, we examine autocorrelations in the level and volatility of the callable bond pricing errors by fitting the following time series models:

\[
\hat{\epsilon}_t = a_0 + a_1 \hat{\epsilon}_{t-1} + a_2 \hat{\epsilon}_{t-2} + a_3 \hat{\epsilon}_{t-3} + \epsilon_t, \\
|\hat{\epsilon}_t| = b_0 + b_1 |\hat{\epsilon}_{t-1}| + b_2 |\hat{\epsilon}_{t-2}| + b_3 |\hat{\epsilon}_{t-3}| + w_t,
\]

where the absolute value of the pricing errors captures the volatility of the pricing errors.\(^{21}\)

\(^{21}\)Since bond yields are characterized by fat tails, it is more efficient to estimate volatility functions using the
Autocorrelations in either the level or volatility of the pricing errors suggest past information can predict future prices, and are evidence of model misspecifications. Panels A and B of Table 7 report the distributions for the estimates and the \( t \)-statistics for the level and volatility of the pricing errors, respectively. All estimates suggest that for most firms, the autocorrelations (at various lags) of the level and volatility of pricing errors are insignificant.

In general, the bond market is viewed as less liquid than the stock market and information that affects firm value may be impounded into stock prices faster than it is in bond prices. As a result, bond pricing errors may depend on lagged stock returns. We next examine the relation between callable bond pricing errors and the firm’s stock returns. We run the following regression model

\[
\hat{\epsilon}_t = c + c_0 R_t + c_1 R_{t-1} + c_2 R_{t-2} + c_3 R_{t-3} + \epsilon_t
\]

where \( c, c_0, c_1, c_2, c_3 \) are constants and \( R_{t-i} \) corresponds to the stock return at time \( t - i \). The stock returns are obtained from the CRSP tape. The regression results in panel A of Table 8 show that for most firms the bond pricing errors are correlated with neither the current nor lagged stock returns. Most estimated coefficients are close to zero and statistically insignificant.

We also examine whether callable bond pricing errors are related to systematic stock market factors. Elton et al. (2001) find that the Fama and French (1993) factors explain a substantial portion of corporate bond spreads. To test whether the Fama-French factors can explain the variation in our bond pricing errors, we run the following regression:

\[
\hat{\epsilon}_t = \gamma_0 + \gamma_1 SMB_t + \gamma_2 HML_t + \gamma_3 RM_t + \epsilon_t,
\]

where \( SMB \) is the return on a portfolio of small stocks minus the return on a portfolio of large stocks, \( HML \) is the return on a portfolio of high book-to-market minus low book-to-market stocks, and \( RM \) is the excess market return. The results of this regression in Panel B of Table 8 show that none of the Fama-French factors help explain the pricing errors for most of the firms.

The above diagnostics investigate some potential model misspecifications. The results show that: (i) our model captures the time series properties of the callable bond pricing errors very well, and (ii) incorporating information from the stock market, both firm-specific or systematic factors, would not help improve model performance.

The results so far show that our model is reasonably successful in explaining the behaviors of call spreads. The nonlinear relation between call spread and interest rate in our model captures to some extent the well-known incentive to call. Moreover, the diagnostics results show that the pricing errors are not due to either liquidity or missing stock market factors. Despite these findings, however, there might still be omitted variables that our model does not capture, and the estimated call intensities may absorb the effects of these omitted variables. This could be a

absolute values of the residuals.
potential concern if one’s main purpose is to infer call intensities. On the other hand, if one’s objective is to get a good price fit, either for trading purposes or price discovery from callable bonds, then empirical evidence suggests that our model serves this end rather well.

5 Conclusion

We develop a reduced-form approach for valuing callable corporate bonds by characterizing the call probability via an intensity process. Asymmetric information and market frictions justify the existence of a call-arrival intensity from the market’s perspective. Our approach extends the reduced-form model of Duffie and Singleton (1999) for defaultable bonds to callable bonds in a manner consistent with differences between call and default decisions. We also provide one of the first comprehensive empirical analyses of callable bonds using both our approach and the traditional approach of valuing callable bonds as bonds with an embedded American option. Empirical results show that the reduced-form model fits callable bond prices well and that it outperforms the traditional approach both in- and out-of-sample.

While our analysis has mainly focused on the pricing of callable bonds, our model has much wider application. For example, our reduced-form model allows researchers to infer a firm’s complicated call decisions from the market prices of callable bonds enabling an examination of both the systematic and firm-specific factors affecting a firm’s ex ante call decision. By decomposing callable bond yields into the default-free interest rate, a credit and a call spread, our multi-factor affine models for callable bond yields facilitates hedging the exposures of callable bonds to different risk factors. Our reduced-form model is also suitable for pricing options on callable bonds, such as the currently popular options on mortgage-backed securities. The reduced-form approach allows us to realistically model the complicated exercising decisions on the prepayment option, which lead to accurate valuation for mortgage options. These possibilities will be considered in future research.

6 Appendix

In the appendix, we first discuss how to obtain a closed-form approximation to callable bond price based on the new method developed in Kimmel (2008). Then we present simulation evidence on the finite sample performances on the extended Kalman filter.
6.1 Closed-form Approximation of Callable Bond Price

To obtain callable bond price, we need to evaluate the following expectation

\[ \pi (s_2, t, T) = E_t^Q \left[ \exp \left( - \int_t^T (1 + \beta_{d2}) s_2, u + \beta c \frac{c}{s_2, u} \right) ds \right]. \tag{A.1} \]

Under certain technical assumptions, the expectation is also a solution to the following partial differential equation

\[ \frac{\partial \pi}{\partial t} = \left[ \kappa_2 \theta_2 - (\kappa_2 + \eta_2) s_2 \right] \frac{\partial \pi}{\partial s_2} + \frac{1}{2} \sigma_2^2 s_2 \frac{\partial^2 \pi}{\partial s_2^2} - \left[ (1 + \beta_{d2}) s_2 + \beta \frac{c}{s_2} \right] \pi, \tag{A.2} \]

with the terminal condition

\[ \pi (s_2, T, T) = 1. \tag{A.3} \]

Since the discount rate, \((1 + \beta_{d2}) s_2 + \beta \frac{c}{s_2}\), is a non-affine function of \(s_2\), closed-form solution to \(\pi (s_2, t, T)\) is not easily available. One way to overcome this difficulty is to approximate \(\pi (s_2, t, T)\) using a power series in \((T - t)\)

\[ \pi (s_2, t, T) = \sum_{i=0}^{\infty} a_i (s_2) \frac{(T - t)^i}{i!}, \tag{A.4} \]

where \(a_0 (s_2) = 1\).

From complex analysis, we know that a power series expansion in \(T - t\) of \(\pi (s_2, T - t)\) converges in a circle on the complex plane that extends to the nearest singularity of \(\pi (s_2, T - t)\). Kimmel (2008) shows that singularities for negative or complex values of \(T - t\) (values not of interest for most applications) can cause power series expansions in \(T - t\) to \(\pi (s_2, T - t)\) to fail to converge on the interval \(T - t \in [0, +\infty)\) (values of \(T - t\) that are of interest). As a result, for large values of \(T - t\) (i.e., for bonds with long maturities), the proposed power series solutions may converge very slowly, or may not converge at all. Kimmel (2008) introduces nonlinear transformations of \(T - t\), denoted as \(\tau\), such that a power series in \(\tau\), instead of \(T - t\), converges for all values of \(T - t \in [0, +\infty)\). Following the approach of Kimmel (2008), we are able to obtain a close-form approximation to callable bond price in power series.\(^{22}\)

Specifically, consider the following transformations proposed by Kimmel (2008),

\[ \tau = \frac{1 - e^{-2b(T - t)}}{2b}, \tag{A.5} \]

\[ z = \frac{2e^{-b(T - t)} \sqrt{s_2}}{\sigma_2}, \tag{A.6} \]

where \(b = \sqrt{(\kappa_2 + \eta_2)^2 + 2(1 + \beta_{d2}) \sigma_2^2}\). Then, the above PDE can be transformed into the general affine form of Kimmel (2008)

\[ \frac{\partial g}{\partial \tau} = \frac{1}{2} \frac{\partial^2 g}{\partial z^2} - \frac{B}{z^2} g \]

\(^{22}\)See Kimmel (2008) for more details.
with the final condition

\[ g(0, z) = \left( \frac{\sigma^2}{2} \right)^{-1} \frac{2\kappa \theta}{\sigma^2} \exp \left( \frac{\kappa^2 + \eta - b}{4} z^2 \right) \]  \hspace{1cm} (A.8)\\

where

\[ B = \frac{3}{8} + \frac{2\kappa^2 \theta^2}{\sigma^2} - \frac{2\kappa \theta_2 + 4\beta c}{\sigma^2}. \]  \hspace{1cm} (A.9)\\

We further expand \( g(0, z) z^{-\frac{\sqrt{1-\beta a}}{z}} \) in a polynomial of \( z^2 \) so that the final condition \( g(0, z) \) takes the following form

\[ \begin{cases} 
  g(0, z) = w_1(z^2) z^{-\frac{1}{\sigma_2^2}} + w_2(z^2) z^{\frac{1}{\sigma_2^2}} & \text{if } \frac{1+\sqrt{1-\beta a}}{2} \notin N, \\
  g(0, z) = w_1(z^2) z^{-\frac{1}{\sigma_2^2}} + w_2(z^2) z^{\frac{1}{\sigma_2^2}} \ln z & \text{if } \frac{1+\sqrt{1-\beta a}}{2} \in N,
\end{cases} \]  \hspace{1cm} (A.10)\\

where \( w_i(\cdot), i = 1, 2 \), is an entire function and

\[ a = \frac{3}{8} - \frac{2\kappa^2 \theta^2}{\sigma^2} - \frac{2\kappa \theta_2 + 4\beta c}{\sigma^2}. \]

Finally, we apply Kimmel’s (2008) method to obtain a power series approximation of \( g(\tau, z) \).

### 6.2 Finite Sample Performances of the Extended Kalman Filter

In this section, we investigate the finite sample performances of the extended Kalman filter in our setting through Monte Carlo simulation. Based on simulated data from the default-free, defaultable, and callable models studied in the paper, we estimate each model using the extended Kalman filter. Our results show that the extended Kalman filter leads to accurate estimates of these models in finite samples.

For the default-free model in (7)-(9), we choose the true parameters to be the same as those estimated from the data in Table 3. We generate 180 monthly observations of a vector of default-free zero-coupon yields from the model. Both the number of observations and the maturities of the yields (0.5, 1, 2, 3, 5, 10, and 30 years) are the same as those used in the empirical study. We also add random observation errors that are normally distributed with a zero mean and a constant variance to the zero-coupon yields. Based on each sample of the contaminated zero-coupon yields, we estimate the default-free model using the extended Kalman filter. We repeat this procedure 500 times and obtain 500 sets of estimates of the parameters. In Panel A of Table A1, we report the true parameters, the mean and the RMSE of the 500 estimates of each parameter. It is clear that the Kalman filter can accurately estimate the default-free model for the sample size considered in our study. Our findings are consistent with that of Lund (1997), Duan and Simonato (1999), and Duffee and Stanton (2004).

For the defaultable model in (10)-(12), we choose the median estimates of the 34 firms used in the empirical analysis as the true parameters (see row 3 of Panel A of Table 4). Based on the same
procedure for the default-free model, we generate 45 monthly prices (the average sample size of the 34 firms) of a defaultable bond with the median coupon rate from the model. To be consistent with the empirical analysis, we assume that both the state variables and the parameters of the default-free model are known when generating the prices of the defaultable bond. We add random observation errors that are normally distributed with a zero mean and a constant variance to the defaultable bond prices. Based on each sample of the contaminated defaultable bond prices, we estimate the defaultable model using extended Kalman filter. We repeat this procedure 500 times and obtain 500 sets of estimates of the parameters. In Panel B of Table A1, we report the true parameters, the mean and the RMSE of the 500 estimates of each parameter. The Kalman filter can accurately estimate the defaultable model for the sample size used in our study.

For the callable model in (13)-(16), we choose the median estimates of the 34 firms used in the empirical analysis as the true parameter (see row 3 of Panel B of Table 4). Based on the same procedure for the defaultable model, we generate 45 monthly prices of a callable bond with the median coupon rate from the model. To be consistent with our empirical analysis, we assume that both the state variables and the parameters of the default-free and defaultable models are known when generating the prices of the callable bond. We also add random observation errors that are normally distributed with a zero mean and a constant variance to the callable bond prices. Based on each sample of the contaminated callable bond prices, we estimate the callable bond model using extended Kalman filter. We repeat this procedure 500 times and obtain 500 sets of estimates of the parameters. In Panel C of Table A1, we report the true parameters, the mean and the RMSE of the 500 estimates of each parameter. The Kalman filter can accurately estimate the callable model for the sample size used in our study. While the observation equations are linear for the default-free model, they are nonlinear for defaultable and callable bonds. Therefore, our results are consistent with that of Trolle and Schwartz (2008), who also consider nonlinear observation equations.
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