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Title: **Default Parameter Estimation using Market Prices**

Mini-Abstract: **A Method for Estimating Recovery Rates and Default Likelihoods Implicit in Debt and Equity Prices**

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Abstract

This paper presents a new methodology for estimating recovery rates and (pseudo) default probabilities implicit in both debt and equity prices. In this methodology, recovery rates and default probabilities are correlated and dependent on the macro-state of the economy. The contribution of this approach is twofold. First, the methodology explicitly incorporates equity prices into the estimation procedure. This inclusion allows the separate identification of both recovery rates and default probabilities, and the use of an expanded and relevant data set. Equity prices may contain a bubble component. This is essential given the recent experience with internet stocks. Second, the methodology explicitly incorporates a liquidity premium into the estimation procedure. This is essential given the large observed variability in the yield spread between risky debt and Treasuries, and the illiquidities present in risky debt markets.

Digest

The available model structures for pricing credit risk can been decomposed into two types: structural and reduced form. Structural models are those that endogenize the bankruptcy process by explicitly modeling the assets and the liability structure of the firm, see Merton (1974). Reduced form models exogenously specify an arbitrage free evolution for the spread between default free and credit risky bonds, see Jarrow and Turnbull (1995), Duffie and Singleton (1999).

Structural models have been successfully implemented in professional software (see Jarrow and Turnbull (2000) for a review). This particular parameterization of the structural approach uses only equity prices and balance sheet data to estimate the bankruptcy process parameters. It is argued that debt markets are too illiquid and debt prices too noisy to be useful, hence, they should be ignored. Unfortunately, this implementation of the structural approach ignores the possibility of stock price bubbles (e.g. internet stocks) and the misspecification that this implies. In contrast, the existing literature on implementing reduced form models concentrates only on debt prices, ignoring equity prices (see Jarrow, Lando, Turnbull (1997), Duffie and Singleton (1999)).

The two approaches seem to have partitioned the market data - structural models use only equity prices and reduced form models use only debt prices. This partitioning is artificial and unnecessary. Both markets provide relevant information about the firm's default process and parameters, and both should be used.

This paper provides a new methodology for implementing reduced form models that includes both debt and equity prices into the estimation procedure. In particular, this paper presents a methodology for estimating recovery rates and (pseudo) default probabilities implicit in debt and equity prices. The methodology is quite general, allowing default probabilities and recovery rates to be correlated and dependent on the macro-state of the economy. This generates a reduced form model that integrates both market and credit risk with correlated defaults.

The contribution of this paper is twofold. First, as previously stated, we provide a methodology that explicitly incorporates equity prices into the reduced form estimation procedure. For a fractional recovery rate process, debt prices only allows the estimation of the expected loss, that is, the multiplicative product of the recovery rate times the (pseudo) default probabilities (see Duffie and Singleton (1999)). In contrast, the introduction of equity prices enables one to separately estimate these quantities. The procedure used to include equity into the reduced form model is one that is commonly employed in the portfolio theory literature (see Duffie (1988)). Simply stated, the equity price is viewed as the present value of future dividends and a resale value. The future resale value is consistent with the existence of equity price bubbles (see Jarrow and Madan (1999)). Given the recent market experience with internet stocks, such an inclusion is necessary for accurate estimation of bankruptcy parameters using equity prices.

Second, debt markets are notoriously illiquid, especially in comparison to equity markets. Our methodology also explicitly incorporates liquidity risk into the reduced form model and the estimation procedure. Liquidity risk is introduced using the notion of a convenience yield, a well-studied concept in the commodities pricing literature that is consistent with an arbitrage-free but incomplete debt market. Liquidity risk introduces an important and necessary additional randomness into the yield spread between risky bond prices and Treasuries. This additional randomness allows for the decomposition of the credit spread into a liquidity risk component and a credit risk component. The liquidity risk adjustment is needed to accurately estimate the bankruptcy parameters from credit spreads.

Default Parameter Estimation using Market Prices

1. Introduction

A Value at Risk measure that successfully integrates market, credit and liquidity risk is the "Holy Grail" of a successful risk management procedure. As I've previously argued (see Jarrow (1998)), the abstract arbitrage-free pricing theory allows this construction, at least conceptually. The remaining obstacles to a successful implementation of this "Holy Grail" are the selection of a particular parameterization of the general model and the estimation of its parameters.

The available model structures have been decomposed into two types: structural and reduced form. Structural models are those that endogenize the bankruptcy process by explicitly modeling the assets and the liability structure of the firm, see Merton (1974). Reduced form models exogenously specify an arbitrage free evolution for the spread between default free and credit risky bonds, see Jarrow and Turnbull (1995), Duffie and Singleton (1999).

Structural models have been successfully implemented in professional software (see Jarrow and Turnbull (2000) for a review). This particular parameterization of the structural approach uses only equity prices and balance sheet data to estimate the bankruptcy process parameters. It is argued that debt markets are too illiquid and debt prices too noisy to be useful, hence, they should be ignored. Unfortunately, this implementation of the structural approach ignores the possibility of stock price bubbles (e.g. internet stocks) and the misspecification that this implies. In contrast, the existing literature on implementing reduced form models concentrates only on debt prices, ignoring equity prices (see Jarrow, Lando, Turnbull (1997), Duffie and Singleton (1999)).

The two approaches seem to have partitioned the market data - structural models use only equity prices and reduced form models use only debt prices. This partitioning is artificial and unnecessary. Both markets provide relevant information about the firm's default process and parameters, and both should be used. The purpose of this paper is to provide a new methodology for implementing reduced form models that includes both debt and equity prices into the estimation procedure. In particular, this paper presents a methodology for estimating recovery rates and (pseudo) default probabilities implicit in debt and equity prices. The methodology is quite general, allowing default probabilities and recovery rates to be correlated and dependent on the macro-state of the economy. This generates a reduced form model that integrates both market and credit risk with correlated defaults.

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An outline of this paper is as follows. Section 2 presents the basics of the model. Section 3 details the risk neutral valuation procedure. The liquidity premium is introduced in section 4. Section 5 constructs a model of the stock price bubble. Section 6 discusses implicit estimation. The general model is illustrated via a practical example in section 7. Section 8 concludes the paper.

2. Model Structure

This section introduces the notation and economic structure of the reduced form model. We assume that there are frictionless markets with no arbitrage opportunities. Markets are not assumed to be complete, perfectly liquid nor are price bubbles excluded.

There is a probability space underlying the economy where P represents the "statistical" or "objective" or "empirical" probability distribution. We will use the term "statistical" probability distribution. The statistical probability distribution is the probability distribution that standard statistical procedures draw inferences about when using historical market prices. Alternatively stated, it is that probability distribution generating the observed debt and equity prices in the economy.

Trading can take place anytime during the interval $[\,0,\ \overline{T}\,]$. Traded are default-free zero-coupon bonds of all maturities, equity, and risky (defaultable) zero-coupon bonds of all maturities. We need some notation to characterize these prices and the subsequent estimation procedure. We now provide this notation.

Let $p(t,T)$ represent the time t price of a default-free dollar paid at time T where $0 \le t \le T \le \overline{T}$. Default-free forward rates $f(t,T)$ are implicitly defined by

$$p(t,T) = e^{-\int_t^T f(t,u)du}. \quad (1)$$

The spot rate of interest is given by $r(t) = f(t,t)$.

The notation for the risky zero-coupon debt prices requires more structure. Consider a firm issuing debt and equity to finance its operations. For the moment, suppose that its debt takes the form of zero-coupon bonds of perhaps different seniority (in the event of default).

Let $v(t,T: i)$ represent the time t price of a promised dollar of seniority i to be paid by this firm at time T where $0 \leq t \leq T \leq \bar{T}$. The debt is risky because if the firm defaults prior to time T, then the promised dollar may not be paid.

Let τ represent the first time that this firm defaults ($\tau > \bar{T}$ is possible if the firm does not default). The default time, τ, is a random variable. We let

$$N(t) = 1_{\{t \geq \tau\}} = \begin{cases} 1 & \text{if} \quad t \geq \tau \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

denote the point process indicating whether or not default has occurred prior to time t. At this stage in the analysis the point process can be a very general stochastic process. We let $\lambda(t)$ represent its random intensity process. The time t intensity process, $\lambda(t)\Delta$, gives the approximate probability of default for this firm over the time interval $[t, t + \Delta]$. [1]

[1] The intensity process is defined under the risk neutral probability. This statement will become clear after section 3 below.

Without loss of generality, if default occurs, we let the zero-coupon bond of seniority i receive a *fractional recovery* of $\delta_i(\tau)v(\tau-, T: i)$ dollars where $0 \le \delta_i(\tau)$ and $\tau-$ represents an instant before default. After default, the debt is worth only a fraction of its pre-default value. The recovery fraction $\delta_i(t)$ is random. At this point, the fractional recovery rate assumption is without loss of generality because the recovery rate process is completely arbitrary. When a specific parameterization for the recovery rate is imposed for empirical estimation, then this assumption becomes restrictive. Note that the recovery rate fraction $\delta_i(t)$ completely specifies the seniority status of the debt issue. The greater the seniority of the debt issue, the larger the recovery rate, everything else constant.

The above formulation is the standard structure imposed in reduced form models. We now turn to the formulation of equity prices. For analysis, it is useful to think of equity as the debt issue of "last" seniority. Equity pays "coupons," called dividends and a liquidating payoff at time T^* for $0 \le T^* \le \overline{T}$.[2] The time t value of these promised payments equals the value of the equity (per share) and is denoted by $\xi(t)$. This is a standard procedure for characterizing equity prices in the portfolio theory literature (see Duffie (1988)). The equity holders receive these payments unless the firm defaults. If default occurs, the equity holders get a

[2] It is convenient to thing of the liquidating payoff as the present value of all future dividends paid over the time period (T^*, ∞).

fractional recovery payment on these promises equal to $\delta_e(\tau)\xi(\tau-)$ where $\delta_e(\tau) \equiv 0$.[3]

We need to develop some notation for these dividend payments. The regular dividends are paid at times $1, 2, \ldots, T^*$ and denoted by D_t at time t. We assume that these dividends are *deterministic* quantities, paid unless the firm defaults prior to the dividend payout date.[4] This formulation implicitly defines T^* as that date under which this deterministic dividend assumption is true. For many stocks, T^* will necessarily be set equal to a year (or less).

The liquidating dividend is paid at time T^* unless default occurs prior to this date. It consists of a random payoff of $L(T^*)$. Let $S(t)$ represent the time t present value of this liquidating dividend, conditional upon no default prior to time t. There is some evidence, for example the recent price growth of internet stocks[5], that stock prices contain a "bubble" or "monetary value" component, see Jarrow and Madan (1999). We let $\theta(t)$ represent this time t bubble component in the stock price.

Given this set-up, it is easy to see[6] that the per share equity value at time t is given by

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[3] In fact, as the subsequent analysis will show, what is really being assumed here is that $\delta_e(\tau)$ is the minimal recovery rate. Under this interpretation, all the subsequent recovery rates will be relative to $\delta_e(\tau)$.

[4] As seen below, if the future dividends are random, then they would be included within the $S(t)$ component.

[5] See Money Magazine April 1999, page 169 for Yahoo's P/E ratio of 1176.6.

[6] This is a simple no arbitrage restriction that the present value of the sum of multiple cash flows equals the sum of the present values of the cash flows.

$$\xi(t) = \begin{cases} S(t) + \theta(t) + \sum_{j \geq t}^{T^*} D_j v(t, j : e) & \text{if} \quad t < \tau \\ 0 & \text{if} \quad t \geq \tau \end{cases} \tag{3}$$

where *v(t, j: e)* represents a zero-coupon bond issued by this firm of seniority e (equity).

The equity value at time *t* is equal to the present value of the liquidating dividend plus a bubble component plus the present value of the regular dividend payments. The present value of the regular dividend payments is seen to be equivalent to a portfolio of risky zero-coupon bonds of a particular seniority. The seniority is that of equity, with a fractional recovery rate of $\delta_e(t)$. If default occurs, then the value of the equity drops to zero because the fractional recovery rate on the dividends (liquidating and regular) is assumed to be zero. The bond's default parameters are explicitly included within this component of the equity's price.

There is a special case of expression (3) worth mentioning because it has been previously used in the option pricing literature.

(Announced Dividend Model) $D_j = 0$ unless $j = t_x$ where t_x is the next ex-dividend date. Let t_a be the announcement date of the next dividend payment.

In this case, expression (3) becomes

$$\xi(t) = \begin{cases} S(t) + \theta(t) + D_{t_x} v(t, t_x : e) & \text{if} \quad t \in [t_a, t_x) \quad \text{and} \quad t < \tau \\ 0 & \text{if} \quad t \in [t_a, t_x) \quad \text{and} \quad t \geq \tau \end{cases} \tag{4.a}$$

and

$$\xi(t) = \begin{cases} S(t) + \theta(t) & if \quad t \notin [t_a, t_x) \quad and \quad t < \tau \\ 0 & if \quad t \notin [t_a, t_x) \quad and \quad t \geq \tau \end{cases} \tag{4.b}$$

The interpretation of expression (4) is that the dividend is known and deterministic only after it is announced (and prior to its payment). This is similar to a model used for the valuation of equity options with a known and discrete dividend, see Jarrow and Turnbull (1996). In this example, the date $T^* = t_x$ but only for $t \geq t_a$; otherwise $T^*=0$. This example clarifies the robustness of the deterministic dividend assumption and the interaction between the definition of T^* and the specification of the regular dividends D_t.

3. Risk Neutral Valuation

This section of the paper presents the valuation formulas used in the estimation of the bankruptcy parameters. Under the assumption of no arbitrage, standard arbitrage pricing theory[7] implies that there exists a probability distribution Q such that present values are computed by discounting at the spot rate of interest and then taking an expectation with respect to Q. For example, using this characterization, we can write

$$p(t,T) = E_t\left(e^{-\int_t^T r(u)du}\right) \tag{5}$$

[7] See Jarrow and Turnbull (1995). No arbitrage guarantees the existence, but not the uniqueness of the probability measure Q. Without any additional hypotheses on the economy, the uniqueness of Q is equivalent to markets being complete, see Battig and Jarrow (1999). In incomplete markets,

where $E_t(.)$ is conditional expectation with respect to Q at time t.

This is the standard risk-neutral pricing relation satisfied by default-free zero-coupon bonds. Applying this valuation methodology to the risky zero-coupon bond prices and the liquidating dividend, we obtain:

$$v(t,T:i) = E_t\left(\delta_i(\tau)v(\tau-,T:i)e^{-\int_t^\tau r(u)du} 1_{(\tau\le T)} + 1\, e^{-\int_t^T r(u)du} 1_{(T<\tau)} \right) \tag{6}$$

$$S(t) = E_t\left(L(T^*)e^{-\int_t^{T^*} r(u)du} 1_{(T^*<\tau)} \right). \tag{7}$$

The risky debt value is composed of two parts. The first is the present value of the promised payment in default. The second is the present value of the promised payment if default does not occur. The present value of the liquidating dividend is similar. The only difference in these two expressions is that $L(T^*)$ is random for the liquidating dividend, while the promised risky debt payment of 1 dollar is not.

There is no analogous expression for the bubble component $\theta(t)$. The reason is that one cannot write the bubble component as a discounted expectation, see Jarrow and Madan (1999).[8]

Using a result from Duffie and Singleton (1999, Theorem 1), under mild conditions[9], we can rewrite expressions (6) and (7) as:

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equilibrium (additional hypotheses) guarantees the uniqueness of Q. The uniqueness of Q is essential for estimation.

$$v(t,T:i) = E_t\left(1\ e^{-\int_t^T [r(u)+\lambda(u)(1-\delta_i(u))]du}\right) \tag{8}$$

$$S(t) = E_t\left(L(T^*)e^{-\int_t^{T^*}[r(u)+\lambda(u)]du}\right) \tag{9}$$

where default has not occurred before or at time t, and $\lambda(u)$ is the intensity process under the risk-neutral measure Q. We call this the pseudo probability of default.

The importance of this simplification cannot be overstated. We see that the risky zero-coupon bond price can again be written as an expected discounted value, but where the discount factor is the spot rate of interest adjusted for the expected loss in default $(\lambda(t)(1 - \delta_i(t)))$. A similar statement applies for the present value of the liquidating (random) dividend.

As pointed out by Duffie and Singleton (1999), the pseudo probability of default always appears in this valuation formula for risky debt as part of a multiplicative product. It is always multiplied by the fractional loss in default $(1 - \delta_i(t))$. Hence, debt prices only allow one to estimate the product of the pseudo probability of default times the fractional loss, and not the pseudo probability of default alone. The introduction of the equity valuation process as in expression

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[8] This insight implies that the techniques for inferring the asset's volatility using Merton's model of risky debt are misspecified in the presence of bubbles.

[9] The condition is that the value of the debt and equity must not jump at the time of default. Given the fractional recovery rate process, this is a reasonable assumption to impose.

(3), in conjunction with expressions (8) and (9), overcomes this difficulty. This is because the fractional loss for equity is known (apriori) and equal to one, i.e., $(1-\delta_e(t)) = 1$ as $\delta_e(t) = 0$.[10] Thus, a joint estimation of pseudo default probabilities and recovery rates, identifying each separately, is possible using both debt and equity prices. A procedure for doing this will be discussed in a subsequent section.

4. The Liquidity Premium

This section includes a liquidity premium into the above model formulation. Liquidity risk is an important consideration in the pricing of risky debt. The importance of its inclusion can be motivated by two observations. First, debt prices are very difficult to obtain due to the sparsity of secondary market trading. In fact, the best frequency of data available (at the writing of this paper) is monthly observations - the Wisconsin data base (see Warga (1995)). Compare this to the readily available frequency in transaction data for equity prices. Second, a study by Schwartz (1998) indicates that even for this monthly bond data, the number of outliers (measured relative to similar debt issues) is significant. One can attribute these outliers to the illiquidity in the market.

Corporate debt issues, analogous to Treasuries, can be used in repurchase agreements as collateral. As such, there are times when particular corporate bonds are in short supply, asking prices are high, and special repo rates are low

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[10] In the event $\delta_e(\tau)$ is not equal zero, then estimated will be the ratio of $(1-\delta_i(\tau))/(1-\delta_e(\tau))$.

(see Rooney (1998)). In these cases, one cannot buy the bond at reasonable prices and liquidity causes bond prices to be "too high". Conversely, in times of credit scares and high market volatilities, corporate (or particular sovereigns, e.g. Russian) bonds can only be sold at discount prices. In these cases, one cannot sell the bond at reasonable prices and liquidity causes bond prices to be "too low".

Although Duffie and Singleton (1999) suggest a modification of expression (8) to incorporate liquidity risk, they do not give a formal argument justifying its inclusion. This section provides such a formal justification based on a related argument used for convenience yields in Treasury securities that is contained in Jarrow and Turnbull (1997). Our justification is consistent with no arbitrage opportunities but an incomplete debt market.

Consider a market where one can not synthetically construct a particular credit risky zero with price $v_l(t,T:i)$. The subscript "*l*" indicates that the market has a liquidity problem. Given an identical credit risky zero with no liquidity problems and price $v(t,T:i)$, the following no arbitrage relationships hold:

$v(t,T:i) \leq v_l(t,T:i)$ when a shortage (and cannot readily buy), and

$v(t,T:i) \geq v_l(t,T:i)$ when a glut (and cannot readily sell).

The argument is simple. When one cannot synthetically construct the bond on the right side of this expression, the act of arbitrage cannot force equality between the two prices. Thus, there exists a function $\gamma_i(t,T)$ such that

$$v_l(t,T:i) = e^{-\gamma_i(t,T)} v(t,T:i). \tag{10}$$

When there is a shortage and one cannot readily buy the risky bond, then $\gamma_i(t,T) \leq 0$ and the function $(-\gamma_i(t,T))$ has the interpretation of being a *positive convenience yield* obtained from holding (storing) the credit risky zero. In this case there are shortages of the risky bond, (special) repo rates are low, and there is a benefit to storing the bond. This is exactly analogous to positive convenience yields associated with storage of other commodities use in production (like oil).

When there is a glut and one cannot readily sell the risky bond, then $\gamma_i(t,T) \geq 0$ the function $(-\gamma_i(t,T))$ has the interpretation of being a *negative convenience yield* obtained from holding (storing) the credit risky zero. In this case, there is a negative externality from holding it in a portfolio. This is an implicit storage cost exactly analogous to the negative convenience yields associated with storage of spoilable commodities.

For equity markets, liquidity costs are assumed to be zero. Here again, as for recovery rates, equity forms the base case against which debt's bankruptcy parameters can be estimated.

5. A Model of the Stock Price Bubble

This section of the paper presents the model for the stock price bubble. For simplicity, we model the bubble component as a random process that is proportional to the present value of the liquidating dividend as in the following expression:

$$\theta(t) = S(t)\left(e^{\int_0^t \mu_\theta(u)du} - 1 \right) \tag{11}$$

where $\mu_\theta(u) \geq 0$ is the continuous return in the stock price due to the bubble component.

Combined, we can rewrite expression (3) as:

$$\xi(t) = \begin{cases} S(t)e^{\int_0^t \mu_\theta(u)du} + \sum_{j \geq t}^{T^*} D_j v(t, j : e) & \text{if } t < \tau \\ 0 & \text{if } t \geq \tau. \end{cases} \tag{12}$$

This represents a convenient decomposition of the stock price into its underlying components.

6. Implicit Estimation

This section studies the joint implicit estimation of the recovery rates and the (pseudo) default probabilities. To do this estimation, additional structure needs to be imposed on both of these quantities. Following Lando (1998), we assume that the default process follows a Cox process where both $\lambda(t)$ and $\delta_i(t)$ are predetermined functions of a vector of observable state variables, represented by $X(t)$ for $0 \leq t \leq \overline{T}$. $X(t)$ is a multi-dimensional stochastic process. The state variables within $X(t)$ could include the spot interest rate, foreign currencies, GNP measures, or a market index.

Formally,

$$\lambda(t) = \lambda(t, X(t)) \quad and \quad \delta_i(t) = \delta_i(t, X(t)). \tag{13}$$

Similarly, the liquidity discount, the bubble component of the stock price, and the present value of the liquidating value of the equity can depend on the same state variables *X(t)* as well, i.e.

$$\begin{aligned}
&\gamma_i(t,T) = \gamma_i(t,T,X(t)), \\
&S(t) = S(t,X(t)), \quad \text{and} \\
&\mu_\theta(t) = \mu_\theta(t,X(t)).
\end{aligned} \tag{14}$$

Of course, prior to estimation, these deterministic functions need to be specified.

The estimation is performed at an arbitrary time t using cross-sectional and time series data. Given is a collection of observable default free and risky zero-coupon bond prices $p(t,T)$ for various *t, T* and $v_l\,(t,T\!: i\,)$ for various *i ,t* and *T.* An observable equity price $\xi\,(t)$, observable (predictable) dividends D_1, ... , D_{T^*}, and observable state variables *X(t).* Note that this is conditioned on the fact that the firm is not yet in default. With these observables, the left side of the following system of equations is determined.

$$v_l(t,T:i) = v_l\big(t,T:i,\lambda(t,X(t)),\delta_i(t,X(t)),\gamma_i(t,T,X(t))\big)$$

for various *i, T* (15a)

$$\xi(t) = \sum_{j \geq t}^{T^*} D_j v(t,j:e,\lambda(t,X(t))) + S(t,X(t))e^{\int_0^t \mu_\theta(u,X(u))du} \tag{15b}$$

In the right side of these equations, we make explicit the dependence of the risky debt and equity prices on the (psuedo) default probabilities (λ), the recovery rate (δ_i), the liquidity premium (γ_i), the bubble component (μ_θ), and the liquidating dividend (*S(t)*). Notice that the equity prices do not depend on a recovery rate or a liquidity premium.

The two systems of equations can be estimated in three stages. Stage one is to estimate the parameters in the system of equations given by expression (15a) for the risky debt prices. This system can be estimated cross-sectionally at a particular time *t*. Here, as long as the number of equations is at least as large as the number of unknowns, the system can be inverted to obtain estimates of the parameters (a sum of squared error minimizing procedure may be necessary). However, as indicated before, the recovery rate and the default probability always appear as a product and are inseparable in this system.

In the above estimation procedure, it was assumed that risky zero-coupon bond prices are observable. This is usually not the case. Instead, risky coupon bond prices are observable. The above procedure can be easily modified to incorporate this difference. This is stage two in the estimation procedure. There are two basic approaches. One is to first strip out the zero-coupon bond prices from the coupon bonds, before applying the above procedure. Various techniques are available in this regard, see Schwartz (1998). The second is to apply the joint estimation procedure directly to the coupon bearing bond prices, using the fact that a risky coupon bond is a portfolio of risky zero-coupon bonds. Hence, in equation (15a), the left side becomes the observable risky coupon bearing bonds while the right side becomes a summation of the relevant zero-coupon bonds weighted by the coupon payments.

Next, stage three is to estimate the parameters in expression (15b) for equity prices. For this estimation, we condition on the fact that default has not yet occurred, i.e. $t < \tau$. This single equation can only be estimated using time series analysis. The unknowns are the (pseudo) default probability, the bubble component and the liquidating dividend. To obtain a solution, at least as many time series observations of $\xi(t)$ as there are unknown parameters in $\lambda(t,X(t)),\ \mu_\theta(t,X(t),\ and\ \ S(t,X(t))$ are needed. Then, given the estimates for the equity price's default parameters, the recovery rates for the different seniority debt issues can be easily inferred from the debt price parameters estimated earlier.

An alternative to the three-stage procedure discussed above is a single stage procedure that jointly estimates all of the parameters from the larger system of equations using both coupon bonds and equity prices together. The difference between the two approaches is that the joint estimation procedure constrains the parameters to be identical across the two markets, while the three-stage procedure does not.

The next section of the paper further describes this estimation procedure for a special case of the above formulation. To estimate the system of equations represented by expression (15), one still needs to specify the various functions in expressions (13) and (14). This is done therein. Without loss of generality, we assume that the risky zero-coupon bond prices are observable.

7. A Practical Empirical Specification

For a practical, yet realistic empirical specification of the reduced form model, we let the state variables *X(t)* describing the system be two: (i) the spot rate of interest and (ii) an general indicator for the health of the economic system - the cumulative excess return on a market index (as measured from some initial date). Higher dimensional systems can also be easily accommodated, but this extension is left to subsequent research.

We need to specify an arbitrage-free evolution for these state variables. First, consider the spot rate of interest *r(t)*. For illustration purposes, we use a single factor model with deterministic volatilities, sometimes called the extended Vasicek model. The term structure evolution is described by the evolution of the spot rate of interest under the risk neutral measure Q.

$$dr(t) = a[\bar{r}(t) - r(t)]dt + \sigma_r dW(t) \tag{16}$$

where $a \neq 0$, $\sigma_r > 0$ are constants, $\bar{r}(t)$ is a deterministic function of t, and *W(t)* is a standard Brownian motion under Q initialized at *W(0) = 0*.

In expression (16), the spot rate of interest follows a mean reverting process under the risk neutral measure. As shown in Heath, Jarrow and Morton (1992), to match an arbitrary initial forward rate curve, one must set

$$\bar{r}(t) = f(0,t) + \left(\partial f(0,t)/\partial t + \sigma_r^2 \left(1 - e^{-2at}\right) / 2a\right) / a \,. \tag{17}$$

Combined, we can rewrite the evolution for the spot rate of interest as:

$$r(t) = f(0,t) + \sigma_r^2 \left(e^{-at} - 1\right)^2 / 2a^2 + \int_0^t \sigma_r e^{-a(t-u)} dW(u) . \quad (18)$$

Note that the spot rate of interest is normally distributed in the extended Vasicek model.

The second state variable is related to a market index, denoted by *M(t)*. The evolution for the market index is assumed to satisfy

$$dM(t) = M(t)\left(r(t)dt + \sigma_m dZ(t)\right) \quad (19)$$

where σ_m is constant, and *Z(t)* is a standard Brownian motion[11] under *Q* initialized at $Z(0) = 0$ correlated with *W(t)* as $dZ(t)dW(t) = \varphi_{rm} dt$ with φ_{rm} a constant.

The market index follows a geometric Brownian motion with drift *r(t)* and volatility σ_m. The drift must be the spot rate of interest under the risk neutral measure. The market index and the spot rate of interest process are correlated, with

$$correlation\ (dM(t) / M(t),\ dr(t)) = \varphi_{rm} dt\ . \quad (20)$$

For subsequent usage, we rewrite the market index process in its integral form.

[11] This assumption implies that the economy is also arbitrage free with respect to inclusion of an additional traded asset, the market index.

$$M(t) = M(0)e^{\int_0^t r(u)du - (\frac{1}{2})\sigma_m^2 t + \sigma_m Z(t)}. \quad (21)$$

Given observation dates *1, 2, 3, ..., t* we can solve expression (21) for *Z(t)* as a function of *Z(t-1)*. This solution is given by

$$Z(t) = Z(t-1) + \left[\log\ M(t)/M(t-1) - \int_{t-1}^{t} r(u)du + \int_{t-1}^{t} (\tfrac{1}{2})\sigma_m^2 du \right] \Big/ \sigma_m \quad (22)$$

$$for\ \ t \geq 1\ \ and\ \ Z(0) = 0.$$

We see here that *Z(t)* is a measure of the cumulative excess return per unit of risk (above the spot rate of interest) on the market index.[12] *Z(t)* becomes our second state variable, chosen because it is normally distributed (as is the spot rate of interest under expression (18)).

We now impose our assumption on the bankruptcy parameters and the recovery rate.

(Spot Rate and Market Index Dependence)

$$\begin{aligned} &\lambda(t) = \lambda_0 + \lambda_1 r(t) + \lambda_2 Z(t) \\ &\delta_i(t) = \delta_i \quad where \\ &\lambda_0, \lambda_1, \lambda_2, \delta_i \quad \text{are constants.} \end{aligned} \quad (23)$$

In this formulation, the (pseudo) probability of default is assumed to be a linear function of the state variables *r(t)* and *Z(t)*. This implies that negative default rates ($\lambda(t) < 0$) are possible. Nonetheless, given the tractability of the subsequent expressions, this is an acceptable first approximation. Its validity awaits empirical

[12] This can be estimated using past observations of *M(t)*.

investigation. Secondly, we assume that the fractional recovery rate is a constant. This also is a first approximation that is easily relaxed.

Given these expressions, it is shown in the appendix that the default free zero-coupon bond and the risky zero-coupon bond's price can be rewritten as:

$$p(t,T) = E_t\left(e^{-\int_t^T r(u)du}\right) = e^{-\mu_1(t,T)+\sigma_1^2(t,T)/2} \tag{24}$$

$$v_1(t,T:i) = e^{-\gamma_i(t,T)} E_t\left(e^{-\int_t^T \left[r(u)+(\lambda_0+\lambda_1 r(u)+\lambda_2 Z(u))(1-\delta_i)\right]du}\right)$$

$$= e^{-\gamma_i(t,T)} p(t,T) e^{-\lambda_0(1-\delta_i)(T-t)-\lambda_1(1-\delta_i)\mu_1(t,T)+\left(2\lambda_1(1-\delta_i)+\lambda_1^2(1-\delta_i)^2\right)\sigma_1^2(t,T)/2} \bullet \tag{25}$$

$$e^{-\lambda_2(1-\delta_i)Z(t)(T-t)+(1+\lambda_1(1-\delta_i))\lambda_2(1-\delta_i)\varphi_{rm}\eta(t,T)+[T-t]^3\lambda_2^2(1-\delta_i)^2/6}$$

where no default has occurred at or prior to time t,

$$\mu_1(t,T) = \int_t^T f(t,u)du + \int_t^T b(u,T)^2 du/2,$$

$$\sigma_1^2(t,T) = \int_t^T b(u,T)^2 du,$$

$$b(u,t) = \sigma_r\left(1-e^{-a(t-u)}\right)/a,$$

$$\eta(t,T) = \int_t^T \int_t^T \left[\int_t^{min(s,u)} \rho(v,s)dv\right]ds\, du = -(\sigma_r/a^3)[1-e^{-a(T-t)}] + (\sigma_r/a^2)e^{-a(T-t)}(T-t) + (\sigma_r/2a)[T-t]^2,$$

and

$$\rho(v,s) = \sigma_r e^{-a(s-v)}.$$

To understand these pricing formulas, one must first recognize that the randomness in their values across time occurs for two reasons. The first reason is due to randomly changing default free rates. This enters through the $\mu_1(t,T)$ term in both expressions (24) and (25), and in particular, through the term involving the current forward rates $\int_t^T f(t,u)du$. The second reason, which only applies to the risky debt, is due to the possibility of default, in which case the risky bond price in expression (25) drops from $v_I(\tau-,T:i)$ to $v_I(\tau,T:i)=\delta_i v_I(\tau-,T:i)$.

To better understand the randomness due to changing default free rates, it is instructive to transform these equations to that contained in Jarrow and Turnbull (2000). In the appendix, it is shown that

$$\mu_I(t,T)=c(t,T)+b(t,T)r(t)/\sigma_r \qquad \text{where} \qquad (26)$$

$$c(t,T)=\int_t^T[f(0,u)+b(0,u)^2/2]du-b(t,T)[f(0,t)+b(0,t)^2/2]/\sigma_r.$$

In this expression, *c(t,T)* and *b(t,T)* are deterministic functions of time. The randomness in expression (26) is due to the spot interest rate *r(t)*. Substituting expression (26) into the pricing expression (24) gives the valuation formula in Jarrow and Turnbull (2000). This substitution also shows that the default-free and risky zero-coupon bond prices are Markov in *r(t).* This Markov structure facilitates computation and it is an advantage of using the extended Vasicek model.

To understand the implications of expression (25) for the yield spread between the risky bond and Treasury prices, we first implicitly define the yield spread at time t for a particular maturity bond T, $\chi(t,T:i)$, as

$$v_l(t,T:i)/p(t,T) = e^{-\chi(t,T:i)(T-t)}. \tag{27}$$

Using expression (26) for $\mu_1(t,T)$ and the definition of the yield spread to Treasuries as given in expression (27), one can show that

$$\begin{aligned}\chi(t,T:i)(T-t) &= \gamma_i(t,T) + \lambda_0(1-\delta_i)(T-t) \\ &+ \lambda_1(1-\delta_i)c(t,T) - (2\lambda_1(1-\delta_i) + \lambda_1{}^2(1-\delta_i)^2)\sigma_1^2(t,T)/2 \\ &+ \lambda_1(1-\delta_i)b(t,T)r(t)/\sigma_r + \lambda_2(1-\delta_i)Z(t)(T-t) \\ &- (1+\lambda_1(1-\delta_i))\lambda_2(1-\delta_i)\varphi_{rm}\eta(t,T) - [T-t]^3\lambda_2^2(1-\delta_i)^2/6.\end{aligned} \tag{28}$$

The yield spread consists of a term due to liquidity risk ($\gamma_i(t,T)$) and a term due to credit risk (all the remaining terms). The yield spread is random because the liquidity risk component $\gamma_i(t,T)$ is random and the credit risk component contains the spot rate of interest *r(t)* and the cumulative excess return on the market index *Z(t)*, both of which are random.

To complete the empirical formulation, we need to specify the functional form for the liquidity discount, the bubble component and the liquidating dividend as given in expression (14). This is the task to which we now turn.

We assume that

$$L(T^*) = L(t)e^{\int_0^t r(u)du - (1/2)\int_0^t \sigma_L^2 du + \int_0^t \sigma_L dw_L(u)} \tag{29}$$

where $\sigma_L > 0$ is a constant, $w_L(t)$ is a Brownian motion under the martingale measure Q with $dZ(t)dw_L(t) = \varphi_{mL}dt$ and $dW(t)dw_L(t) = \varphi_{rL}dt$ where φ_{rL}, φ_{mL} are constants.

In expression (15), *L(t)* represents the time t liquidation value of the firm's assets less liabilities. This liquidation value can be viewed as the market value of a portfolio containing the firm's assets and liabilities. This portfolio's value, if held by a default free entity, would evolve through time according to expression (29). For simplicity, this evolution is assumed to be a geometric Brownian motion under the martingale measure with a drift rate equal to the spot rate *r(u)*. The value of this portfolio at time *T** would be *L(T*)* with probability one.

In our case, however, *L(t)* is held by a firm that can default prior to time *T**. If the firm defaults, due to bankruptcy costs (lawyer's fees, lost sales, etc.), the liquidation value declines by the fraction $(1-\delta_e)$. This implies that the present value of the liquidation value to an equity holder in the risky firm is less than or equal to the present value of the underlying portfolio of assets/liabilities to a default free agent, i.e.

$$S(t)1_{\{\tau>t\}} = E_t\left(L(T^*)e^{-\int_t^{T^*}[r(u)+\lambda(u)]du} \right) \leq E_t\left(L(T^*)e^{-\int_t^{T^*}r(u)du} \right) = L(t)\,. \qquad (30)$$

Here, it can be shown that:

$$S(t) = \frac{L(t)}{p(t,T^*)} e^{-\lambda_1\sigma_1^2(t,T^*)-\lambda_1\sigma_L\varphi_{rL}\int_t^{T^*}b(u,T^*)du-\lambda_2\varphi_{rm}\eta(t,T^*)-\lambda_2\sigma_L\varphi_{mL}(T^*-t)^2/2} v(t,T^*:e)\,. \quad (31)$$

The proof is in the appendix. Unfortunately, this expression for the present value of the liquidating dividend has the unknown *L(t)* on the right side of expression (31). This form of the present value expression can provide no additional inference about the default parameter process $\lambda(t)$ as there are more unknowns than observables. This insight motivates the following transformation of expression (15b).

Let Δ correspond to a discrete change in time. Taking logarithms of expression (31) and subtracting time $t-\Delta$ from time *t* gives:

$$log\left(\left[\xi(t)-\sum_{j\geq t}^{T^*} D_j v(t,j:e)\right] \Big/ \left[\xi(t-\Delta)-\sum_{j\geq t-\Delta}^{T^*} D_j v(t-\Delta,j:e)\right]\right)=log\big(L(t)/L(t-\Delta)\big)$$
$$+\int_{t-\Delta}^{t}\mu_\theta(u)du+log\big(\psi(t,T^*)/\psi(t-\Delta,T^*)\big)+log\big(v(t,T^*:e)/v(t-\Delta,T^*:e)\big) \qquad (32)$$

where

$$\psi(t,T^*)=-\lambda_1\sigma_1^2(t,T^*)-\lambda_1\sigma_L\varphi_{rL}\int_t^{T^*}b(u,T^*)du-\lambda_2\varphi_{rm}\eta(t,T^*)-\lambda_2\sigma_L\varphi_{mL}(T^*-t)^2/2\,.$$

Next, using the evolution of the liquidation value *L(t)* as given in expression (29) we obtain

$$log\big(L(t)/L(t-\Delta)\big)=\int_{t-\Delta}^{t}r(u)du-(1/2)\sigma_L^2\Delta+\sigma_L[w_L(t)-w_L(t-\Delta)]\,. \qquad (33)$$

This evolution is under the martingale measure. Using Girsanov's theorem, we can change the to the statistical probability measure. The change transforms the original Brownian motion to a new Brownian motion under the statistical measure and an adjustment for a risk premium,

$$w_L(t) = \hat{w}_L(t) + \int_0^t \Theta_L(u)du \tag{34}$$

where $\hat{w}_L(t)$ is a Brownian motion under the statistical measure $\hat{Q}$ and $\Theta_L(u)$ is the liquidation value's risk premium.

Using this change of measure gives:

$$log\big(L(t)/L(t-\Delta)\big) = \int_{t-\Delta}^{t} [r(u) + \sigma_L \Theta_L(u)]du - (1/2)\sigma_L^2 \Delta + \varepsilon(t-\Delta) \tag{35}$$

where the error terms $\varepsilon(t-\Delta) \equiv \sigma_L [\hat{w}_L(t) - \hat{w}_L(t-\Delta)]$ for all t are independent, identically and normally distributed with zero mean and variance $\sigma_L^2 \Delta$.

Substitution of expression (34) into (32) yields:

$$\begin{aligned} &log\left(\left[\xi(t) - \sum_{j\geq t}^{T^*} D_j v(t, j:e)\right] \Big/ \left[\xi(t-\Delta) - \sum_{j\geq t-\Delta}^{T^*} D_j v(t-\Delta, j:e)\right]\right) - \int_{t-\Delta}^{t} r(u)du = \\ &\int_{t-\Delta}^{t} [\sigma_L \Theta_L(u) - (1/2)\sigma_L^2 + \mu_\theta(u)]du + log\big(\psi(t,T^*)/\psi(t-\Delta,T^*)\big) \\ &\quad + log\big(v(t,T^*:e)/v(t-\Delta,T^*:e)\big) + \varepsilon(t-\Delta). \end{aligned} \tag{36}$$

For estimation purposes, the excess return on equity can be written as:

$$log\left(\frac{\left[\xi(t) - \sum_{j\geq t}^{T^*} D_j v(t, j:e)\right]}{\left[\xi(t-\Delta) - \sum_{j\geq t-\Delta}^{T^*} D_j v(t-\Delta, j:e)\right]}\right) - r(t-\Delta)\Delta$$

$$
\begin{aligned}
&\approx -\lambda_0 \Delta - \lambda_1 \left(\left(\frac{b(t-\Delta,T^*)^2}{2} \right) \Delta - log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)} \right) \right) + \lambda_1^2 \left(\frac{b(t-\Delta,T^*)^2}{2} \right) \Delta \\
&\quad - \lambda_1 \sigma_L \varphi_{rL} b(t-\Delta,T^*) \Delta - \lambda_2 [Z(t)(T^*-t) - Z(t-\Delta)(T^*-t-\Delta)] \\
&\quad + \lambda_2^2 (T^*-t)^2 \Delta / 3 - \lambda_2 \sigma_L \varphi_{mL} (T^*-t) \Delta \\
&\quad + [\sigma_L \gamma_L (t-\Delta, X(t-\Delta)) - (1/2)\sigma_L^2 + \mu_\theta (t-\Delta, X(t-\Delta))] \Delta \\
&\quad + \varepsilon(t-\Delta).
\end{aligned}
\tag{37}
$$

The proof is in the appendix. This is a non-linear regression equation for the excess return on the equity where the coefficients give the default parameters. To complete this estimation a model for the risk premium and the bubble component is required. For example, if one assumes that the risk premium can be approximated using a capital asset pricing model and that the bubble component can be approximated with the variance of the stock price,

$$
\begin{aligned}
&[\sigma_L \gamma_L (t-\Delta, X(t-\Delta)) - (1/2)\sigma_L^2 + \mu_\theta (t-\Delta, X(t-\Delta))] \Delta \\
&\qquad = \beta_0 \, log(M(t)/M(t-\Delta)) + \beta_1 \, log(\xi^2(t)/\xi^2(t-\Delta))
\end{aligned}
\tag{38}
$$

then the system is easily estimated with only 2 extra parameters (β_0, β_1).

Finally, one can model the liquidity discount as a first order Taylor series approximation for a more general function of the state variables:

$$
\gamma_i(t,T) = \gamma_0^i + \gamma_1^i r(t) + \gamma_2^i Z(t) . \tag{39}
$$

Combining all of the above empirical specifications into expression (15) gives the following system of equations. This system contains both cross sectional and time series observations.

$$
\begin{aligned}
v_1(t,T:i) = {} & e^{-\gamma_0^i+\gamma_1^i r(t)+\gamma_2^i Z(t)}\, p(t,T) \bullet \\
& e^{-\lambda_0(1-\delta_i)(T-t)-\lambda_1(1-\delta_i)\mu_1(t,T)+\left(2\lambda_1(1-\delta_i)+\lambda_1^2(1-\delta_i)^2\right)\sigma_1^2(t,T)/2} \bullet \\
& e^{-\lambda_2(1-\delta_i)Z(t)(T-t)+(1+\lambda_1(1-\delta_i))\lambda_2(1-\delta_i)\varphi_{rm}\eta(t,T)+[T-t]^3\lambda_2^2(1-\delta_i)^2/6}
\end{aligned}
\tag{40.a}
$$

for various i,T, and t, and

$$
\begin{aligned}
& log\left(\frac{\left[\xi(t) - \sum_{j\geq t}^{T^*} D_j v(t,j:e)\right]}{\left[\xi(t-\Delta) - \sum_{j\geq t-\Delta}^{T^*} D_j v(t-\Delta,j:e)\right]}\right) - r(t-\Delta)\Delta \;\approx\; -\lambda_0\Delta \\
& -\lambda_1\left(\left(\frac{b(t-\Delta,T^*)^2}{2}\right)\Delta - log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right)\right) + \lambda_1^2\left(\frac{b(t-\Delta,T^*)^2}{2}\right)\Delta \\
& -\lambda_1\sigma_L\varphi_{rL} b(t-\Delta,T^*)\Delta - \lambda_2[Z(t)(T^*-t) - Z(t-\Delta)(T^*-t-\Delta)] \\
& +\lambda_2^2(T^*-t)^2\Delta/3 - \lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta \\
& +\beta_0\, log(M(t)/M(t-\Delta)) + \beta_1\, log(\xi^2(t)/\xi^2(t-\Delta)) \\
& +\varepsilon(t-\Delta)
\end{aligned}
\tag{40.b}
$$

for various t.

Expression (40.a) is the debt pricing equation while expression (40.b) is the equity pricing equation. The solution to this system of equations can be obtained via a non-linear regression. The solution depends on the initial forward rate curve $\{f(0,T)\}$, the term structure evolution parameters $\{a, \sigma_r\}$, the market index parameters $\{\varphi_{rm}, \sigma_m\}$, and the liquidating dividend parameters $(\varphi_{rL}, \varphi_{mL}, \sigma_L)$. Additional parameters to be estimated are the default process coefficients ($\lambda_0, \lambda_1, \lambda_2$), the recovery rate ($\delta_i$), the liquidity discount coefficients ($\gamma_0^i, \gamma_1^i, \gamma_2^i$), and the bubble/ risk premium coefficients (β_0, β_1). This system of equations needs to be estimated using both cross sectional and time series data.

8. Conclusion

This paper develops a new procedure for implicit estimation of a liquidity premium, the recovery rate and the (pseudo) default probabilities using debt and equity prices. This new procedure is quite general. It allows the default process to be correlated across firms and dependent on the macro-state of the economy. It allows debt markets to be illiquid and equity markets to contain bubbles. Its empirical evaluation, however, awaits subsequent research.

Although the procedure described herein formally estimates the pseudo or risk neutral default intensities, if the reduced form model is properly specified, there is good reason to believe that the statistical and risk neutral default intensity functions are equal. This occurs if default risk is idiosyncratic, after properly conditioning on macro-state variables (see Jarrow, Lando, Yu (1999)). This idiosyncratic default risk hypothesis is intuitively plausible and its validation awaits subsequent research. The above methodology is consistent with this properly conditioned, reduced form model.

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Appendix

Derivation of expressions (24) and (25)

From expression (21) we have that

$$r(s) = f(t,s) + b(t,s)^2/2 + \int_t^s \rho(v,s)dW(v)$$

where $\rho(v,s) = \sigma_r e^{-a(s-v)}$ and

$$b(t,s) = \int_t^s \rho(t,v)dv = \sigma_r(1-e^{-a(s-t)})/a\,.$$

Define $X_1 \equiv \int_t^T r(s)ds = \int_t^T f(t,s)ds + \int_t^T b(t,s)^2\, ds/2 + \int_t^T \int_t^s \rho(v,s)dW(v)ds\,.$

Changing the order of integration and a direct computation yields

$$\int_t^T b(t,s)^2\, ds/2 = \int_t^T b(v,T)^2\, dv/2 \quad and$$

$$\int_t^T \int_t^s \rho(v,s)dW(v)ds = \int_t^T b(v,T)dW(v).$$

Substitution gives

$$\int_t^T r(s)ds = \int_t^T f(t,s)ds + \int_t^T b(v,T)^2\, dv/2 + \int_t^T b(v,T)dW(v).$$

A direct computation gives

$$\mu_1(t,T) \equiv E_t(\int_t^T r(s)ds) = \int_t^T f(t,s)ds + \int_t^T b(v,T)^2\, dv/2 \qquad \text{and}$$

$$\sigma_1^2(t,T) \equiv Var_t(\int_t^T r(s)ds) = \int_t^T b(v,T)^2\, dv.$$

Define $X_2 \equiv \int_t^T Z(u)du$. Following Parzen (1962, p. 81),

$$\mu_2(t,T) \equiv E_t(X_2) = \int_t^T E_t(Z(u))du = \int_t^T Z(t)du = Z(t)[T-t]$$

$$\sigma_2^2(t,T) \equiv Var_t(X_2) = 2\int_t^T \int_t^v (u-t)dudv = [T-t]^3/3$$

$$\sigma_{12}(t,T) \equiv Cov_t(X_1, X_2) = E_t(X_1 X_2) - E_t(X_1)E_t(X_2)$$

$$= E_t\left(\int_t^T r(s)ds \int_t^T Z(s)ds\right) - \left(\int_t^T f(t,s)ds + \int_t^T b(v,T)^2\, dv/2\right)Z(t)[T-t]$$

$$= \int_t^T \int_t^T E_t(r(s)Z(u))dsdu - \left(\int_t^T f(t,s)ds + \int_t^T b(v,T)^2\, dv/2\right)Z(t)[T-t].$$

But,

$$E_t(r(s)Z(u)) =$$

$$E_t\left[\left(f(t,s) + b(t,s)^2/2 + \int_t^s \rho(v,s)dW(v)\right)\left(\int_t^u dZ(v) + Z(t)\right)\right] =$$

$$\varphi_{rm} \int_0^{min(s,u)} \rho(v,s)dv + \left[f(t,s) + b(t,s)^2/2\right]Z(t).$$

Now,

$$\int_t^T \int_t^T \left[f(t,s) + b(t,s)^2/2\right]Z(t)duds = Z(t)[T-t]\int_t^T \left[f(t,s) + b(t,s)^2/2\right]ds$$

$$= Z(t)[T-t]\left(\int_t^T f(t,s)ds + \int_t^T b(v,T)^2\, dv/2\right).$$

Substitution and simplification yields

$$\sigma_{12}(t,T) = \varphi_{rm}\left[\int_t^T \left(\int_t^T \left(\int_t^{min(s,u)} \rho(v,s)dv\right)ds\right)du\right].$$

Given (X_1, X_2) is bivariate normal, we have (see Hogg and Craig (1970)).

$$E_t\left[e^{AX_1 + BX_2}\right] = e^{\mu_1 A + \mu_2 B + \left[\sigma_1^2 A^2 + 2\sigma_{12}AB + \sigma_2^2 B^2\right]/2} \tag{A1}$$

where

$$\mu_1 \equiv E_t(X_1), \mu_2 \equiv E_t(X_2), \sigma_1^2 \equiv Var_t(X_1), \sigma_2^2 \equiv Var_t(X_2) \quad and \quad \sigma_{12} \equiv Cov_t(X_1, X_2).$$

Then, for expression (24), using (A1) we get:

$$E_t\left[e^{-[1+\lambda_1(1-\delta_i)]\int_t^T r(u)du}\right] = E_t\left[e^{AX_1}\right] = e^{\mu_1 A + \sigma_1^2 A^2/2}$$

where $A = -[1 + \lambda_1(1-\delta_i)]$. This gives the desired result.

For expression (25),

$$E_t\left[e^{-[1+\lambda_1(1-\delta_i)]\int_t^T r(u)du-\lambda_2(1-\delta_i)\int_t^T Z(u)du}\right]=E_t\left[e^{AX_1+BX_2}\right] \quad where$$

$$A=-[1+\lambda_1[1-\delta_i]] \quad and$$

$$B=-\lambda_2(1-\delta_i).$$

(A1) gives the desired result.

Derivation of Expression (26)

Under the spot rate model of expression (18), it can be shown that the arbitrage-free forward rate process is given by

$$f(t,u)=f(0,u)+\int_0^t \alpha(v,u)dv+\int_0^t \rho(v,u)dW(v)$$

where $\alpha(v,u)=\rho(v,u)\int_v^u \rho(v,s)ds$.

We will be interested in evaluating the following integral of forward rates:

$$\int_t^T f(t,u)du=\int_t^T f(0,u)du+\int_t^T\left(\int_0^t \alpha(v,u)dv\right)du+\int_t^T\left(\int_0^t \rho(v,u)dW(v)\right)du.$$

The following facts can be proven by direct computation given the definitions of $\alpha(v,u),\rho(v,u),b(v,u)$.

$$\int_t^u \alpha(v,u)dv=b(t,u)^2/2 \qquad \text{and}$$

$$\int_t^T\left(\int_0^t \rho(v,u)dW(v)\right)du=b(t,T)\int_0^t \rho(v,t)dW(v).$$

Using the first of these facts, one can show that:

$$\int_t^T\left(\int_0^t \alpha(v,u)dv\right)du = \int_t^T\left(\int_0^u \alpha(v,u)dv\right)du - \int_t^T\left(\int_t^u \alpha(v,u)dv\right)du$$
$$= \int_t^T\left(b(0,u)^2/2\right)du - \int_t^T\left(b(t,u)^2/2\right)du.$$

But, we know from expression (18) that

$$r(t) - f(0,t) - b(0,t)^2/2 = \int_0^t \rho(v,t)dW(v).$$

Using this observation and the second fact stated above gives:

$$\int_t^T\left(\int_0^t \rho(v,u)dW(v)\right)du = b(t,T)\left(r(t) - f(0,t) - b(0,t)^2/2\right)/\sigma_r.$$

Direct substitution of these observations gives that

$$\int_t^T f(t,u)du = \int_t^T f(0,u)du + \int_t^T\left(b(0,u)^2/2\right)du - \int_t^T\left(b(t,u)^2/2\right)du$$
$$+ b(t,T)\left(r(t) - f(0,t) - b(0,t)^2/2\right)/\sigma_r.$$

Substitution of this integral into the definition of $\mu_1(t,T)$ along with the fact that

$$\int_t^T b(t,u)^2\, du/2 = \int_t^T b(u,T)^2\, du/2$$

gives expression (26).

Equity Model Computations

From expression (9) and (29) we have that

$$S_t = E_t\left(L(T^*)e^{-\int_t^{T^*}[r(u)+\lambda(u)]du}\right) = L(t)e^{-(1/2)\int_t^{T^*}\sigma_L^2 du}E_t\left(e^{-\int_t^{T^*}\lambda(u)du + \int_t^{T^*}\sigma_L dw_L(u)}\right).$$

Next, using expression (23) gives

$S_t = L(t)e^{-(1/2)\int_t^{T^*}\sigma_L^2 du - \int_t^{T^*}\lambda_0 du}E_t\left(e^{-\int_t^{T^*}\lambda_1 r(u)du - \int_t^{T^*}\lambda_2 Z(u)du + \int_t^{T^*}dw_L(u)}\right)$. To evaluate the expectation, we use (A.1) with the following identifications.

$A \equiv 1,\; x \equiv -\lambda_1 \int_t^{T^*} r(u)du - \lambda_2 \int_t^{T^*} Z(u)du = -\lambda_1 X_1 - \lambda_2 X_2,\; B \equiv 1,\; y \equiv \int_t^{T^*} \sigma_L dw_L(u)$.

The expectation is

$e^{\mu_x + \mu_y + (1/2)\sigma_x^2 + \sigma_{xy} + (1/2)\sigma_y^2}$ where

$$\mu_x = -\lambda_1 \mu_1(t,T^*) - \lambda_2 Z(t)(T^* - t);\; \mu_y = 0;$$
$$\sigma_x^2 = \lambda_1^2 \sigma_1^2(t,T^*) + 2\lambda_1\lambda_2\sigma_{12}(t,T^*) + \lambda_2^2(T^* - t)^3/3;\; \sigma_y^2 = \sigma_L^2(T^* - t);$$
$$\sigma_{xy} = cov_t\left(-\lambda_1 \int_t^{T^*} r(u)du - \lambda_2 \int_t^{T^*} Z(u)du, \int_t^{T^*} \sigma_L dw_L(u)\right)$$
$$= -\lambda_1 \sigma_L\, cov_t\left(\int_t^{T^*} r(u)du, \int_t^{T^*} dw_L(u)\right) - \lambda_2 \sigma_L\, cov_t\left(\int_t^{T^*} Z(u)du, \int_t^{T^*} dw_L(u)\right)$$

But,

$$cov_t\left(\int_t^{T^*} r(u)du, \int_t^{T^*} dw_L(u)\right) = cov_t\left(\int_t^{T^*} b(u,T^*)dW(u), \int_t^{T^*} dw_L(u)\right) = \varphi_{rL} \int_t^{T^*} b(u,T^*)du.$$

Next,

$$cov_t\left(\int_t^{T^*} Z(u)du, \int_t^{T^*} dw_L(u)\right) = E_t\left(\int_t^{T^*} Z(u)du \int_t^{T^*} dw_L(u)\right)$$ since $E_t\left(\int_t^{T^*} dw_L(u)\right) = 0$.

But, $E_t\left(\int_t^{T^*} Z(u)du \int_t^{T^*} dw_L(u)\right) = E_t\left(\int_t^{T^*} [Z(u) - Z(t) + Z(t)]du \int_t^{T^*} dw_L(u)\right)$

$= E_t\left(\int_t^{T^*} E_u\left(\int_t^u dZ(v) \int_t^{T^*} dw_L(v)\right)du\right) = E_t\left(\int_t^{T^*}\left(\int_t^u dZ(v) \int_t^u dw_L(v)\right)du\right)$ since

$E_u\left(\int_u^{T^*} dw_L(u)\right) = 0$. Finally,

$$E_t\left(\int_t^{T^*}\left(\int_t^u dZ(v) \int_t^u dw_L(v)\right)du\right) = \left(\int_t^{T^*} E_t\left(\int_t^u dZ(v) \int_t^u dw_L(v)\right)du\right) = \int_t^{T^*} \varphi_{mL}(u - t)du$$

Thus, $cov_t\left(\int_t^{T^*} Z(u)du, \int_t^{T^*} dw_L(u)\right) = \varphi_{mL}(T^* - t)^2/2$.

Substitution of these results into the expression for *S(t)* gives:

$$S_t = L(t) e^{-\lambda_0(T^*-t) - \lambda_1\mu_1(t,T^*) - \lambda_2 Z(t)(T^*-t) + (1/2)\lambda_1^2\sigma_1^2(t,T^*) + \lambda_1\lambda_2\sigma_{12}(t,T^*) + \lambda_2^2(T^*-t)^3/6} \bullet$$
$$e^{-\lambda_1\sigma_L\varphi_{rL}\int_t^{T^*} b(u,T^*)du - \lambda_2\sigma_L\varphi_{mL}(T^*-t)^2/2 + (1/2)\sigma_L^2(T^*-t)} \qquad (\#)$$

From expression (25) we have that:

$$\frac{v(t,T^*:E)}{p(t,T^*)}e^{-\lambda_1\sigma_1^2(t,T^*)-\lambda_2\sigma_{12}(t,T^*)} =$$

$$e^{-\lambda_0(T^*-t)-\lambda_1\mu_1(t,T^*)-\lambda_2 Z(t)(T^*-t)+(1/2)\lambda_1^2\sigma_1^2(t,T^*)+\lambda_1\lambda_2\sigma_{12}(t,T^*)+\lambda_2^2(T^*-t)^3/6}.$$

Substitution gives

$$S_t = L(t)\frac{v(t,T^*:E)}{p(t,T^*)}e^{-\lambda_1\sigma_1^2(t,T^*)-\lambda_2\sigma_{12}(t,T^*)-\lambda_1\sigma_L\varphi_{rL}\int_t^{T^*} b(u,T^*)du-\lambda_2\sigma_L\varphi_{mL}(T^*-t)^2/2}.$$

Next, we derive the expression for the excess return on equity. From expression (12) and (#) above, taking natural logarithms and then the difference from time $t-\Delta$ to time t we obtain:

$$\log\left(\frac{\xi_t-\sum_{j\geq t}^{T^*}D_j v(t,j:e)}{\xi_{t-\Delta}-\sum_{j\geq t-\Delta}^{T^*}D_j v(t-\Delta,j:e)}\right)=\log\left(\frac{L_t}{L_{t-\Delta}}\right)+\int_{t-\Delta}^{t}\mu_\theta(u)du-\lambda_0[(T^*-t)-(T^*-t-\Delta)]$$
$$-\lambda_1[\mu_1(t,T^*)-\mu_1(t-\Delta,T^*)]-\lambda_2[Z(t)(T^*-t)-Z(t-\Delta)(T^*-t-\Delta)]$$
$$+(1/2)\lambda_1^2[\sigma_1^2(t,T^*)-\sigma_1^2(t-\Delta,T^*)]+\lambda_1\lambda_2[\sigma_{12}(t,T^*)-\sigma_{12}(t-\Delta,T^*)]$$
$$+\lambda_2^2[(T^*-t)^3-(T^*-t-\Delta)^3]/6-\lambda_1\sigma_L\varphi_{rL}[\int_t^{T^*} b(u,T^*)du-\int_{t-\Delta}^{T^*} b(u,T^*)du]$$
$$-\lambda_2\sigma_L\varphi_{mL}[(T^*-t)^2-(T^*-t-\Delta)^2]/2.$$

In the following identifications, terms of order Δ^p for $p\geq 2$ are omitted.

$\lambda_0[(T^*-t)-(T^*-t-\Delta)]=\lambda_0\Delta$. Next, we have:

$$\lambda_1[\mu_1(t,T^*)-\mu_1(t-\Delta,T^*)]=\lambda_1[-\log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right)+\int_{t-\Delta}^{t}b(v,T^*)^2dv/2]$$
$$\approx\lambda_1[-\log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right)+b(t-\Delta,T^*)^2\Delta/2].$$

$$(1/2)\lambda_1^2[\sigma_1^2(t,T^*)-\sigma_1^2(t-\Delta,T^*)]=(1/2)\lambda_1^2\int_{t-\Delta}^{t}b(v,T^*)^2dv$$
$$\approx b(t-\Delta,T^*)^2\Delta/2.$$

$$\lambda_1\lambda_2[\sigma_{12}(t,T^*)-\sigma_{12}(t-\Delta,T^*)]=$$

$$\lambda_1\lambda_2\varphi_{rm}[\int_t^T\int_t^T\int_t^{min(s,u)}\rho(v,s)dvdsdu-\int_{t-\Delta}^T\int_{t-\Delta}^T\int_{t-\Delta}^{min(s,u)}\rho(v,s)dvdsdu]$$

$$\leq\lambda_1\lambda_2\varphi_{rm}[\int_t^T\int_t^T\int_{t-\Delta}^{min(s,u)}\rho(v,s)dvdsdu-\int_{t-\Delta}^T\int_{t-\Delta}^T\int_{t-\Delta}^{min(s,u)}\rho(v,s)dvdsdu]$$

$$=\lambda_1\lambda_2\varphi_{rm}\int_{t-\Delta}^t\int_{t-\Delta}^t\int_{t-\Delta}^{min(s,u)}\rho(v,s)dvdsdu\leq\lambda_1\lambda_2\varphi_{rm}\int_{t-\Delta}^t\int_{t-\Delta}^t\int_{t-\Delta}^t\rho(v,s)dvdsdu$$

$$\approx\lambda_1\lambda_2\varphi_{rm}\rho(t-\Delta,t)\Delta^3.$$

$$\lambda_2^2[(T^*-t)^3-(T^*-t-\Delta)^3]/6=\lambda_2^2[2(T^*-t)^2\Delta-(T^*-t)^2\Delta^2-\Delta^3]/6$$

$$\approx\lambda_2^2(T^*-t)^2\Delta/3.$$

$$\lambda_1\sigma_L\varphi_{rL}[\int_t^{T^*}b(u,T^*)du-\int_{t-\Delta}^{T^*}b(u,T^*)du]=\lambda_1\sigma_L\varphi_{rL}\int_{t-\Delta}^t b(u,T^*)du$$

$$\approx\lambda_1\sigma_L\varphi_{rL}b(t-\Delta,T^*)\Delta.$$

$$\lambda_2\sigma_L\varphi_{mL}[(T^*-t)^2-(T^*-t-\Delta)^2]/2=\lambda_2\sigma_L\varphi_{mL}[2(T^*-t)\Delta-\Delta^2]/2$$

$$\approx\lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta.$$

Combined,

$$log\left(\frac{\xi_t-\sum_{j\geq t}^{T^*}D_j v(t,j:e)}{\xi_{t-\Delta}-\sum_{j\geq t-\Delta}^{T^*}D_j v(t-\Delta,j:e)}\right)=log\left(\frac{L_t}{L_{t-\Delta}}\right)+\int_{t-\Delta}^t\mu_\theta(u)du-\lambda_0\Delta$$

$$-\lambda_1[-log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right)+b(t-\Delta,T^*)^2\Delta/2]$$

$$-\lambda_2[Z(t)(T^*-t)-Z(t-\Delta)(T^*-t-\Delta)]$$

$$+\lambda_1^2 b(t-\Delta,T^*)^2\Delta/2+\lambda_2^2(T^*-t)^2\Delta/3-\lambda_1\sigma_L\varphi_{rL}b(t-\Delta,T^*)\Delta$$

$$-\lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta.$$

This completes the derivation.