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Journal of Banking & Finance 24 (2000) 271–299

Journal of
BANKING &
FINANCE

www.elsevier.com/locate/econbase

The intersection of market and credit risk [☆]

Robert A. Jarrow ^{a,1}, Stuart M. Turnbull ^{b,*}

^a *Johnston Graduate School of Management, Cornell University, Ithaca, New York, USA*

^b *Canadian Imperial Bank of Commerce, Global Analytics, Market Risk Management Division,
BCE Place, Level 11, 161 Bay Street, Toronto, Ont., Canada M5J 2S8*

Abstract

Economic theory tells us that market and credit risks are intrinsically related to each other and not separable. We describe the two main approaches to pricing credit risky instruments: the structural approach and the reduced form approach. It is argued that the standard approaches to credit risk management – CreditMetrics, CreditRisk+ and KMV – are of limited value when applied to portfolios of interest rate sensitive instruments and in measuring market and credit risk.

Empirically returns on high yield bonds have a higher correlation with equity index returns and a lower correlation with Treasury bond index returns than do low yield bonds. Also, macro economic variables appear to influence the aggregate rate of business failures. The CreditMetrics, CreditRisk+ and KMV methodologies cannot reproduce these empirical observations given their constant interest rate assumption. However, we can incorporate these empirical observations into the reduced form of Jarrow and Turnbull (1995b). Drawing the analogy. Risk 5, 63–70 model. Here default probabilities are correlated due to their dependence on common economic factors. Default risk and recovery rate uncertainty may not be the sole determinants of the credit spread. We show how to incorporate a convenience yield as one of the determinants of the credit spread.

For credit risk management, the time horizon is typically one year or longer. This has two important implications, since the standard approximations do not apply over a one

[☆] The views expressed in this paper are those of the authors and do not necessarily reflect the position of the Canadian Imperial Bank of Commerce.

* Corresponding author. Tel.: +1-416-956-6973; fax: +1-416-594-8528.

E-mail address: turnbust@cibc.ca (S.M. Turnbull).

¹ Tel.: +1-607-255-4729.

year horizon. First, we must use pricing models for risk management. Some practitioners have taken a different approach than academics in the pricing of credit risky bonds. In the event of default, a bond holder is legally entitled to accrued interest plus principal. We discuss the implications of this fact for pricing. Second, it is necessary to keep track of two probability measures: the martingale probability for pricing and the natural probability for value-at-risk. We discuss the benefits of keeping track of these two measures. © 2000 Elsevier Science B.V. All rights reserved.

JEL classification: G28; G33; G2

Keywords: Credit risk modeling; Pricing; Default probabilities

1. Introduction

In the current regulatory environment, the BIS (1996) requirements for specific risk specify that “concentration risk”, “spread risk”, “downgrade risk” and “default risk” must be “appropriately” captured. The principle focus of the recent Federal Reserve Systems Task Force Report (1998) on Internal Credit Risk Models is the allocation of economic capital for credit risk, which is assumed to be separable from other risks such as market risk. Economic theory tells us that market and credit risk are intrinsically related to each other and, more importantly, they are not separable. If the market value of the firm’s assets unexpectedly changes – generating market risk – this affects the probability of default – generating credit risk. Conversely, if the probability of default unexpectedly changes – generating credit risk – this affects the market value of the firm – generating market risk.

The lack of separability between market and credit risk affects the determination of economic capital, which is of central importance to regulators. It also affects the risk adjusted return on capital used in measuring the performance of different groups within a bank.² Its omission is a serious limitation of the existing approaches to quantifying credit risk.

The modern approach to default risk and the valuation of contingent claims, such as debt, starts with the work of Merton (1974). Since then, Merton’s model, termed the *structural approach*, has been extended in many different ways. Unfortunately, implementing the structural approach faces significant practical difficulties due to the lack of observable market data on the firm’s value. To circumvent these difficulties, Jarrow and Turnbull (1995a, b) infer the conditional martingale probabilities of default from the term structure of credit spreads. In the Jarrow–Turnbull approach, termed the *reduced form approach*,

² For an introduction to risk adjusted return on capital, see Crouhy et al. (1999).

market and credit risk are inherently inter-related. These two approaches are described in Section 2.

CreditMetrics, CreditRisk+ and KMV have become the standard methodologies for credit risk management. The CreditMetrics and KMV methodologies are based on the structural approach, and the CreditRisk+ methodology originates from an actuarial approach to mortality.

The KMV methodology has many advantages. First, by relying on the market value of equity to estimate the firm's volatility, it incorporates market information on default probabilities. Second, the graph relating the *distance to default* to the observed default frequency implies that the estimates are less dependent on the underlying distributional assumptions. There are also a number of disadvantages.

Many of the basic inputs to the KMV model – the value of the firm, the volatility and the expected value of the rate of return on the firm's assets – cannot be directly observed. Implicit estimation techniques must be used and there is no way to check the accuracy of the estimates. Second, interest rates are assumed to be deterministic. While this assumption probably has little effect on the estimated default probability over a one year horizon, it limits the usefulness of the KMV methodology when applied to loans and other interest rate sensitive instruments. Third, an implication of the KMV option model is that as the maturity of a credit risky bond tends to zero, the credit spread also tends to zero. Empirically, we do not observe this implication. Fourth, historical data are used to determine the expected default frequency and consequently there is the implicit assumption of stationarity. This assumption is probably not valid. For example, in a recession, the true curve may shift upwards implying that for a given distance to default, the expected default frequency has increased. Consequently, the KMV methodology underestimates the true probability of default. The reverse occurs if the economy is experiencing strong economic growth. Finally, an ad hoc and questionable liability structure for a firm is used in order to apply the option theory.

CreditMetrics represents one of the first publicly available attempts using probability transition matrices to develop a portfolio credit risk management framework that measures the marginal impact of individual bonds on the risk and return of the portfolio. The CreditMetrics methodology has a number of limitations. First, it considers only credit events because the term structure of default free interest rates is assumed to be fixed. CreditMetrics assumes no market risk over a specified period. Although this is reasonable for floating rate and short dated notes, it is less reasonable for zero-coupon bonds, and inaccurate for CLOs, CMOs, and derivative transactions. Second, the CreditMetrics default probabilities do not depend upon the state of the economy. This is inconsistent with the empirical evidence and with current credit practices. Third, the correlation between asset returns is assumed to equal the correlation between equity returns. This is a crude approximation given

uncertain bond returns. The CreditMetrics outputs are sensitive to this assumption.

A key difficulty in the structural-based approaches of KMV and CreditMetrics is that they must estimate the correlation between the rates of return on assets using equity returns, as asset returns are unobservable. Initial results suggest that the credit VARs produced by these methodologies are sensitive to the correlation coefficients on asset returns and that small errors are important.³ Unfortunately, because asset returns cannot be observed, there is no direct way to check the accuracy of these methodologies.

The CreditRisk+ methodology has some advantages. First, CreditRisk+ has closed form expressions for the probability distribution of portfolio loan losses. Thus, the methodology does not require simulation and computation is relatively quick. Second, the methodology requires minimal data inputs of each loan: the probability of default and the loss given default. No information is required about the term structure of interest rates or probability transition matrices. However, there are a number of disadvantages.

First, CreditRisk+ ignores the stochastic term structure of interest rates that affect credit exposure over time. Exposures in CreditRisk+ are predetermined constants. The problems with ignoring interest rate risk made in the previous section on CreditMetrics are also pertinent here. Second, even in its most general form where the probability of default depends upon several stochastic factors, no attempt is made to relate these factors to how exposure changes. Third, the CreditRisk+ methodology ignores non-linear products such as options, or even foreign currency swaps.

Practitioners and regulators often calculate VAR measures for credit and market risk separately and then add the two numbers together. This is justified by arguing that it is difficult to estimate the correlation between market and credit risk. Therefore, to be conservative assume perfect correlation, compute the separate VARs and then add. This argument is simple and unsatisfactory.

It is not clear what is meant by the statement that market risk and credit risk are perfectly correlated. There is not one but many factors that affect market risk exposure, the probability of default and the recovery rate. These factors have different correlations, which may be positive or negative. If the additive methodology suggested by regulators is conservative, how conservative? Risk capital under the BIS 1988 Accord was itself viewed as conservative. Excessive capital may be inappropriately required. By not having a model that explicitly incorporates the effects of credit risk upon price, it is not clear that market risk itself is being correctly estimated. For example, if the event of default is modeled by a jump process and defaults are correlated, then it is well known

³ See Crouhy and Mark (1998).

that the standard form of the capital asset pricing model used for risk management is mis-specified.⁴

Another criticism voiced by regulators is that we do not have enough data to test credit models. “A credit event (read default) is a rare event. Therefore we need data extending over many years. These data do not exist and therefore we should not allow credit models to be used for risk management.”⁵ This is a narrow perspective. For markets where there is sufficient data to construct term structures of credit spreads, we can test credit models such as the reduced form model described in Section 4, using the same criteria as for testing market risk models. Since the testing procedures for market risk are well accepted, this nullifies this criticism raised by regulators.

We briefly review the empirical research examining the determinants of credit spreads in Section 3. It is empirically observed that returns on high yield bonds have a higher correlation with equity index returns and a lower correlation with Treasury bond index returns than do low yield bonds. The KMV and CreditMetrics methodologies are inconsistent with these empirical observations due to their assumption of constant interest rates. Altman (1983/1990) and Wilson (1997a, b) show that macro-economic variables affect the aggregate number of business failure.

In Section 4 we show how to incorporate these empirical findings into the reduced form model of Jarrow and Turnbull. This is done by modeling the default process as a multi-factor Cox process; that is, the intensity function is assumed to depend upon different state variables. This structure facilitates using the volatility of credit spreads to determine the factor inputs. In a Cox process, default probabilities are correlated due to their dependence upon the same economic factors. Because default risk and an uncertain recovery rate may not be the sole determinants of the credit spread, we show how to incorporate a convenience yield as an additional determinant. This incorporates a type of liquidity risk into the estimation procedure.

Another issue relating to credit risk in VAR computations is the selection of the time horizon. For market risk management in the BIS 1988 Accord and the 1996 Amendment, time horizons are typically quite short – 10 days – allowing the use of delta–gamma–theta–approximations. For credit risk management time horizons are typically much longer than 10 days. A liquidation horizon of one year is quite common. This has two important implications. First, it implies that the pricing approximations used for market risk management are inadequate. It is necessary to employ

⁴ See Jarrow and Rosenfeld (1984).

⁵ This view is repeated in the recent Basle report: “Credit Risk Modelling” (1998).

exact valuation models because second order Taylor series expansions leave too much error.

In the academic literature it is often assumed that the recovery value of a bond holder's claim is proportional to the value of the bond just prior to default. This is a convenient mathematical assumption. Courts, at least in the United States, recognize that bond holders can claim accrued interest plus the face value of the bond in the event of default. This is a different recovery rate structure. The legal approach is often preferred by industry participants. In Section 4 we show how to extend the existing credit risk models to incorporate these different recovery rate assumptions.

The second issue in credit risk model implementation is that it is necessary to keep track of two distinct probability measures. One is the natural or empirical measure. For pricing derivative securities, this natural probability measure is changed to the martingale measure (the so-called "risk-neutral" distribution). For risk management it is necessary to use both distributions. The martingale distribution is necessary to value the instruments in the portfolio. The natural probability distribution is necessary to calculate value-at-risk. We clarify this distinction in the text. We also show that we can infer the market's assessment of the probability of default under the natural measure. This provides a check on the estimates generated by Moody's, Standard and Poor's and KMV.

A summary is provided in Section 5.

2. Pricing credit risky instruments

This section describes the two approaches to credit risk modeling – the structural and reduced form approaches. The first approach – see Merton (1974) – relates default to the underlying assets of the firm. This approach is termed the structural approach. The second approach – see Jarrow and Turnbull (1995a,b) – prices credit derivatives off the observable term structures of interest rates for the different credit classes. This approach is termed the reduced form approach.

2.1. Structural approach

The structural approach is best exemplified by Merton (1974, 1977), who considers a firm with a simple capital structure. The firm issues one type of debt – a zero-coupon bond with a face value F and maturity T . At maturity, if the value of the firm's assets is greater than the amount owed to the debt holders – the face amount F – then the equity holders pay off the debt holders and retain the firm. If the value of the firm's assets is less than the face value, the equity holders default on their obligations. There are no costs associated with default

and the absolute priority rule is obeyed. In this case, debt holders take over the firm and the value of equity is zero, assuming limited liability.⁶

In this simple framework, Merton shows that the value of risky debt, $v_1(t, T)$, is given by

$$v_1(t, T) = FB(t, T) - p[V(t)], \quad (2.1)$$

where $B(t, T)$ is the time t value of a zero-coupon bond that pays one dollar for sure at time T , $V(t)$ is the time t value of the firm's assets, and $p[V(t)]$ is the value of a European put option⁷ on the assets of the firm that matures at time T with a strike price of F .

To derive an explicit valuation formula, Merton imposed a number of additional assumptions. First, the term structure of interest rates is deterministic and flat. Second, the probability distribution of the firm's assets is described by a lognormal probability distribution. Third, the firm is assumed to pay no dividends over the life of the debt. In addition, the standard assumptions about perfect capital markets apply.⁸

The Merton model has at least five implications. First, when the put option is deep out-of-the-money ($V(t) \gg F$), the probability of default is low and corporate debt trades as if it is default free. Second, if the put option trades in-the-money, the volatility of the corporate debt is sensitive to the volatility of the underlying asset.⁹ Third, if the default free interest rate increases, the spread associated with corporate debt decreases.¹⁰ Intuitively, if the default free spot interest rate increases, keeping the value of the firm constant, the mean of the asset's probability distribution increases and the probability of default declines. As the market value of the corporate debt increases, the yield-to-maturity decreases, and the spread declines. The magnitude of this change is larger the higher the yield on the debt. Fourth, market and credit risk are not separable. To see this, suppose that the value of the firm's assets unexpectedly decreases, giving rise to market risk. The decrease in the asset's value increases the probability of default, giving rise to credit risk. The converse is also true. This interaction of market and credit risk is discussed in Crouhy et al. (1998). Fifth, as the maturity of the zero-coupon bond tends to zero, the credit spread also tends to zero.

⁶ See Halpern et al. (1980).

⁷ For an introduction to the pricing of options, see Jarrow and Turnbull (1996b).

⁸ These assumptions are described in detail in Jarrow and Turnbull (1996b, p. 34)

⁹ Using put-call parity, expression (2.1) can be written $v_1(t, T) = V(t) - c[V(t)]$, where $c[V(t)]$ is the value of a European call option with strike price F and maturing at time T . If $V(t) \ll F$ then $c[V(t)]$ is 'small' and $v_1(t, T)$ is trading like unlevered equity.

¹⁰ Let $v_1(0, T) \equiv FB(0, T) \exp(-S_p T)$, where S_p denotes the spread. Then $\partial S_p / \partial r = -(V(0)/v_1(0, T))N(-d_1) \leq 0$, where $d_1 \equiv \{\ln V(0)/FB(0, T) + \sigma^2 T/2\}/\sigma\sqrt{T}$, $N(\cdot)$ is the cumulative normal distribution function, and r is the free interest rate.

There are at least four practical limitations to implementing the Merton model. First, to use the pricing formulae, it is necessary to know the market value of the firm's assets. This is rarely possible as the typical firm has numerous complex debt contracts outstanding traded on an infrequent basis. Second, it is also necessary to estimate the return volatility of the firm's assets. Given that market prices cannot be observed for the firm's assets, the rate of return cannot be measured and volatilities cannot be computed. Third, most corporations have complex liability structures. In the Merton framework, it is necessary to simultaneously price all the different types of liabilities senior to the corporate debt under consideration. This generates significant computational difficulties.¹¹ Fourth, default can only occur at the time of a coupon and/or principal payment. But in practice, payments to other liabilities other than those explicitly modeled may trigger default.

Nielson et al. (1993) and Longstaff and Schwartz (1995a, b) take an alternative route in an attempt to avoid some of these practical limitations. In their approach, capital structure is assumed to be irrelevant. Bankruptcy can occur at any time and it occurs when an identical but unlevered firm's value hits some exogenous boundary. In default the firm's debt pays off some fixed fractional amount. Again the issue of measuring the return volatility of the firm's assets must be addressed.¹² In order to facilitate the derivation of 'closed' form solutions, interest rates are assumed to follow an Ornstein–Uhlenbeck process. Unfortunately, Cathcart and El-Jahel (1998) demonstrate that for long-term bonds the assumption of normally distributed interest rates, implicit in an Ornstein–Uhlenbeck process, can cause problems. Cathcart and El-Jahel assume a square root process with parameters suitably chosen to rule out negative rates.¹³ However, they impose an additional assumption which implies that spreads are independent of changes in the underlying default free term structure, contrary to empirical observation.¹⁴

2.2. *Reduced form approach*

One of the earliest examples of the reduced form approach is Jarrow and Turnbull (1995b). Jarrow and Turnbull (1995b) allocate firms to credit risk classes.¹⁵ Default is modeled as a point process. Over the interval $(t, t + \Delta t]$ the

¹¹ See Jones et al. (1984).

¹² See Wei and Guo (1997) for an empirical comparison of the Merton and Longstaff and Schwartz models.

¹³ Cathcart and El-Jahel formulate the model in terms of a 'signaling variable.' They never identify this variable and offer no hint of how to apply their model in practice.

¹⁴ Kim et al. (1993) assume a square root process for the spot interest rate that is correlated with the return on assets.

¹⁵ See Litterman and Iben (1991).

default probability conditional upon no default prior to time t is approximately $\lambda(t)\Delta t$ where $\lambda(t)$ is the intensity (hazard) function. Using the term structure of credit spreads for each credit class, they infer the expected loss over $(t, t + \Delta t]$, that is the product of the conditional probability of default and the recovery rate under the equivalent martingale (the ‘risk neutral’) measure. In essence, they use observable market data – credit spreads – to infer the market’s assessment of the bankruptcy process and then price credit risk derivatives.

In the simple numerical examples contained in Jarrow and Turnbull (1995a, b, 1996a,b), stochastic changes in the credit spread only occur if default occurs. To model the volatility of credit spreads, a more detailed specification is required for the intensity function and/or the recovery function. Das and Tufano (1996) keep the intensity function deterministic and assume that the recovery rate is correlated with the default free spot rate. Das and Tufano assume that the recovery rate depends upon state variables in the economy and is subject to idiosyncratic variation. The interest rate proxies the state variable. Monkkonen (1997) generalizes the Das and Tufano model by allowing the probability of default to depend upon the default free rate of interest. He develops an efficient algorithm for inferring the martingale probabilities of default.

The formulation in Jarrow and Turnbull (1995b) is quite general and allows for the intensity (hazard) function to be an arbitrary stochastic process. Lando (1994/1997) assumes that the intensity function depends upon different state variables. This is referred to as a Cox process. Roughly speaking, a Cox process when conditioned on the state variables acts like a Poisson process. Lando (1994/1997) derives a simple representation for the valuation of credit risk derivatives.

Lando derives three results. First, consider a contingent claim that pays some random amount X at time T provided default has not occurred, zero otherwise. The time t value of the contingent claim is

$$\begin{aligned} E_t^Q \left[\exp \left(- \int_t^T r(s) ds \right) X 1(\Gamma > T) \right] \\ = 1(\Gamma > t) E_t^Q \left[\exp \left(- \int_t^T r(s) + \lambda(s) ds \right) X \right], \end{aligned} \quad (2.2)$$

where $r(t)$ is the instantaneous spot default free rate of interest, Γ denotes the random time when default occurs and $1(\Gamma > t)$ is an indicator function that equals 1 if default has not occurred by time t , zero otherwise. The superscript Q is used to denote the equivalent martingale measure. Expression (2.2) represents the expected discounted payoff where the discount rate ($r(s) + \lambda(s)$) is adjusted for the default probability. Similar expressions can be obtained for alternative payoff structures.

Second, consider a security that pays a cash flow $Y(s)$ per unit time at time s provided default has not occurred, zero otherwise. The time t value of the security is

$$\begin{aligned}
 E_t^Q \left[\int_t^T Y(s) 1(\Gamma > s) \exp \left(- \int_t^s r(u) du \right) ds \right] \\
 = 1(\Gamma > t) E_t^Q \left[\int_t^T Y(s) \exp \left(- \int_t^s r(u) + \lambda(u) du \right) ds \right]. \tag{2.3}
 \end{aligned}$$

Third, consider a security that pays $Z(\Gamma)$ if default occurs at time Γ , zero otherwise. The time t value of the security is

$$\begin{aligned}
 E_t^Q \left[\exp \left(- \int_t^T r(s) ds \right) Z(\Gamma) \right] \\
 = 1(\Gamma > t) E_t^Q \left[\int_t^T Z(s) \lambda(s) \exp \left(- \int_t^s r(u) + \lambda(u) du \right) ds \right]. \tag{2.4}
 \end{aligned}$$

The specification of the recovery rate process is an important component in the reduced form approach. In the Jarrow and Turnbull (1995a, b) model, it is assumed that if default occurs on, say, a zero-coupon bond, the bond holder will receive a known fraction of the bond’s face value at the maturity date. To determine the present value of the bond in the event of default, the default free term structure is used. Alternatively, Duffie and Singleton (1998) assume that in default the value of the bond is equal to some fraction of the bond’s value just prior to default. This assumption allows Duffie and Singleton to derive an intuitively simple representation for the value of a risky bond. For example, the value of a zero-coupon risky bond paying a promised dollar at time T is

$$v(t, T) = 1(\Gamma > t) E_t^Q \left[\exp \left(- \int_t^T r(s) + \lambda(s) L(s) ds \right) \right], \tag{2.5}$$

where the loss function $L(t) \equiv 1 - \delta(t)$ and $\delta(t)$ is the recovery rate function. Hughston (1997) shows that the same result can be derived in the J–T framework.¹⁶

Modeling the intensity function as a Cox process allows us to model the empirical observations of Duffee (1998), Das and Tufano (1996) and Shane (1994) that the credit spread depends on both the default free term structure and an equity index. The work of Jarrow and Turnbull (1995a, b), Duffie and Singleton (1998), Hughston (1997) and Lando (1994/1997) implies that for many credit derivatives we need only model the expected loss, that is the product of the intensity function and the loss function.

¹⁶ This also implies that we can interpret the work of Ramaswamy and Sundaresan (1986) as an application of this theory.

For valuing credit derivatives whose payoffs depend on credit rating changes, Jarrow et al. (1997) describe a simple model that explicitly incorporates a firm's credit rating as an indicator of default. This model can also be used for risk management purposes as it is possible to price portfolios of corporate bonds and credit derivatives in a consistent fashion. Interestingly, the CreditMetrics methodology described in Section 4 of this paper can be viewed as a special case of the JLT model, where there is no interest rate risk.

3. Empirical evidence

There is considerable empirical evidence consistent with changes in credit spreads and changes in default free interest rates being negatively correlated. Duffee (1998) fits a regression of the form

$$\Delta\text{Spread}_t = b_0 + b_1\Delta Y_t + b_2\Delta\text{Term}_t + e_t$$

using monthly corporate bond data from the period January 1985 to March 1995, where Spread_t is the spread at time t for a bond maturing at time T , ΔSpread_t the change in the spread from t to $t + 1$ keeping maturity T fixed, ΔY_t denotes the change in the three month Treasury yield, Term_t denotes the difference between the 30 year constant Treasury bond yield and the three month Treasury bill yield, ΔTerm_t denotes the change in Term over the period, t to $t + 1$ and e_t denotes a zero mean unit variance random term. The estimated coefficients, b_1 and b_2 , are negative and increase in absolute magnitude as the credit quality decreases irrespective of maturity. Similar results are also reported by Das and Tufano (1996).¹⁷

Longstaff and Schwartz (1995a,b) using annual data from 1977 to 1992 fit a regression of the form

$$\Delta\text{Spread}_t = b_0 + b_1\Delta\text{Yield}_t + b_2I_t + e_t,$$

where ΔYield_t denotes the change in the 30 year Treasury, I_t denotes the return on the appropriate equity index and e_t denotes a zero mean unit variance random term. For credit classes Aaa, Aa, A, and Baa industrials, the estimated coefficients are negative.¹⁸ Irrespective of maturity, the coefficients b_1 and b_2 increase in absolute magnitude as the credit quality decreases. However, the

¹⁷ Das and Tufano used monthly data for the period 1971–1991. It is not clear if they filtered their data to eliminate bonds with optionality.

¹⁸ The estimated negative coefficients are not surprising, given the work of Merton (1974). An increase in the Treasury bill rate increases the expected rate of return on a firm's assets, and hence lowers the probability of default. This increases the price of the risky debt and lowers its yield. An increase in the index proxies for an increase in the values of the firm's assets. This lowers the probability of default and hence the yield on the risky debt.

Longstaff and Schwartz results must be treated cautiously as their data include bonds with embedded options. This caution is justified by the work of Duffee (1998) who shows that this can have a major impact on regression results.

Shane (1994) using monthly data over the period 1982–1992 found that returns on high yield bonds have a higher correlation with the return on an equity index than low yield bonds and a lower correlation with the return on a Treasury bond index than low yield bonds. It is not reported whether Shane filtered her data to eliminate bonds with embedded options.

Wilson (1997a, b) examined the effects of macro-economic variables – GDP growth rate, unemployment rate, long-term interest rates, foreign exchange rates and aggregate saving rates – in estimating default rates. While the *R*-squares are impressive, the explanatory importance of the macro-economics variables is debatable. If an economic variable has explanatory power, then a change in the variable should cause a change in the default rate, provided the explanatory variables are not co-integrated. To examine this, an estimation based on changes in variables is needed. Unfortunately, Wilson does the estimation using only levels.

Altman (1983/1990) uses first order differences, the explanatory variables being the percentage change in real GNP, percentage change in the money supply, percentage change in the Standard & Poor index and the percentage change in new business formation. Altman finds a negative relation between changes in these variables and changes in the aggregate number of business failures. Not surprisingly, the reported *R*-squares are substantially lower than those reported in Wilson.

All of these studies suggest that credit spreads are affected by common economic underlying influences¹⁹. We show in the next section how to incorporate these empirical findings using the reduced form model of Jarrow and Turnbull.

4. The reduced form model of Jarrow and Turnbull

The CreditMetrics, CreditRisk+ and KMV methodologies do not consider both market and credit risk. These methodologies assume interest rates are constant and consequently they cannot value derivative products that are sensitive to interest rate changes, such as bonds and swaps. In this section we show how to incorporate both market and credit risk into the reduced form model of Jarrow and Turnbull (1995a, b) in a fashion consistent with the empirical findings discussed in the last section. Following Lando (1994/1997), we model the intensity function as a multi-factor Cox process. One can use the

¹⁹ See Pedrosa and Roll (1998) for further evidence.

volatility of credit spreads to estimate the sensitivity of the intensity function to these different factors. We will also discuss the question of correlation and its role in the Jarrow–Turnbull model.

The typical time horizon used for credit risk models is one year. This is justified on the basis of the time necessary to liquidate a portfolio of credit risky instruments. The relatively long time horizon implies that we cannot use the approximations employed in market risk management where the time horizon is typically of the order of 10 days. Consequently we need to use for risk management the same models that are used for pricing. Here practitioners have gone a slightly different route than academics. Duffie and Singleton (1998) assume that in the event of default an instrument's value is proportional to its value just prior to default. In actuality, courts in the United States recognize that in the event of default, bond holders can claim accrued interest plus the face value of the bond. This different recovery rate structure is often used by practitioners in the pricing of credit sensitive instruments. We examine its implications for the valuation of coupon bonds.

Default risk and recovery rate uncertainty may not be the sole determinants of the credit spread. Liquidity risk may also be an important component. Practitioners, when applying reduced form models such as the Jarrow–Turnbull, often use LIBOR instead of the Treasury curve in an attempt to mitigate such difficulties. We show how to incorporate a convenience yield in the determination of the credit spread.

A second consequence of the longer time horizon employed in credit risk management is the need to keep track of two probability measures: the natural and martingale. For pricing derivatives, the martingale measure is used (the so-called risk-neutral distribution). For risk management it is necessary to use both distributions. The natural measure is used in the determination of VAR. At the end of the specified time horizon, it is necessary to value the instruments in the portfolio and this again requires the use of the martingale distribution.

4.1. Two factor model

We know from the work of Altman (1983/1990) and Wilson (1997a, b) that macro-economic factors have explanatory power in predicting the number of defaults. We also know that high yield bonds have a higher correlation with the return on an equity index and a lower correlation with the return on a Treasury bond index than do low yield bonds. One can incorporate these correlations into the probability of default $\lambda(t)\Delta t$ over the interval $(t, t + \Delta t]$. To describe the dependence of the probability of default on the state of the economy, we use two proxy variables: the spot interest rate and the unexpected change in the market index. Changes in the default free spot interest rate and the market index are readily observable on a daily basis, unlike many macro-economic variables that are only reported quarterly.

Let $I(t)$ denote a market index such as the Standard and Poor 500 stock index. Under the equivalent martingale measure Q it is assumed that changes in the index are described by a geometric Brownian motion

$$dI(t) = [r(t)dt + \sigma_I dW_I(t)]I(t), \quad (4.1)$$

where σ_I is the return volatility on the index and $W_I(t)$ is a Brownian motion. Let $y(t) = \ln[I(t)]$ so that

$$dy = (r(t) - \sigma_I^2/2)dt + \sigma_I dW_I(t)$$

and

$$y(t) - y(0) = \int_0^t [r(s) - \sigma_I^2/2] ds + \sigma_I \int_0^t dW_I(s). \quad (4.2)$$

For tractability, we assume that the intensity function is of the form

$$\lambda(t) = a_0(t) + a_1 r(t) + \beta \sigma_I W_I(t), \quad (4.3)$$

where a_1 and β are constants, and $a_0(t)$ is a deterministic function that can be used to calibrate the model to the observed term structure. The coefficient a_1 measures the sensitivity of the intensity function to the level of interest rates, and β measures the sensitivity to the cumulative unanticipated changes in the market index. The assumption of normality allows the derivation of closed form solutions, such as expression (4.5) below. One of the disadvantages of this assumption is that the intensity function can be negative. In lattice-based models, this difficulty can be avoided via the use of non-linear transformations – see Jarrow and Turnbull (1997a).²⁰

We assume that the instantaneous default free forward rates are normally distributed:

$$df(t, T) = \sigma^2 \exp[-\vartheta(T-t)]b(t, T)dt + \sigma \exp[-\vartheta(T-t)]dW_1(t) \quad (4.4)$$

under the equivalent martingale measure Q – see Heath et al. (1992) – where $b(t, T) \equiv \{1 - \exp[-\vartheta(T-t)]\}/\vartheta$. If $\vartheta = 0$, $b(t, T) = T - t$. The parameter ϑ is often referred to as a mean reversion or a volatility reduction factor (see Jarrow and Turnbull (1996a, ch. 16) for a more detailed discussion). This assumption implies that the spot interest rate is normally distributed.

Under this assumption, the value of a credit risky zero-coupon bond is given by

²⁰ It is sometimes argued that when considering a long dated bond, one should replace the spot rate with a long dated yield. To the extent that the spot interest rate measures the state of the economy over the life of the bond, expression (4.3) is appropriate. In a multi-factor model of the term structure, as described in Heath et al. (1992), then the spot interest rate is not sufficient. For many applications, however, a one factor model will suffice.

$$v(t, T) = 1(\Gamma > t)B(t, T) \exp \{ -a_3 b(t, T)[r(t) - f(0, t)] - \beta_1(T - t)W_I(t) + c_2(t, T) \} \tag{4.5}$$

where $a_3 \equiv a_1 L$, $\beta_2 \equiv \sigma_1 L$, and L is the constant loss rate,

$$c_2(t, T) \equiv c_1(t, T) + \frac{1}{2} \beta_1^2 \int_t^T (T - u)^2 du + a_2 \beta_1 \rho \int_t^T b(u, T)(T - u) du,$$

$$c_1(t, T) \equiv -L \int_t^T a_0(u) du + a_3(\sigma^2/2) \left[b(0, t)^2 b(t, T) - \int_t^T b(0, u)^2 du \right] - a_3 \int_t^T f(0, u) du + (\sigma^2/2)(2a_3 + a_3^2) \int_t^T b(s, T)^2 ds,$$

and ρ is the correlation coefficient between changes in the index and the term structure. A proof is given in Appendix A. Expression (4.5) has an intuitive reformulation.

Let $\chi(t, T)$ denote the credit spread defined implicitly by the expression

$$v(t, T) = 1(\Gamma > t)B(t, T) \exp[-\chi(t, T)(T - t)].$$

Using expression (4.5), this implies that

$$\chi(t, T)(T - t) = a_3 b(t, T)[r(t) - f(0, t)] + \beta_1(T - t)W_I(t) - c_2(t, T). \tag{4.6}$$

We see that changes in the level of interest rates and unanticipated changes in the market index affect the credit spread. The volatility of the spread, ignoring the event of default, is given by

$$\sigma_\chi(t, T)(T - t) = \left\{ a_3^2 b(t, T)^2 \sigma^2 + \beta_1^2 (T - t)^2 + 2a_3 b(t, T) \beta_1 (T - t) \sigma \rho \right\}^{1/2}. \tag{4.7}$$

The credit spread can be used to estimate the parameters a_3 and β_1 in expression (4.6). Given these parameters, the function $\{a_0(t)\}$ can be used to calibrate the initial term structure of credit spreads.

Expression (4.6) can alternatively be written in the form

$$\chi(t, T)(T - t) = a_1 L y_T(t)(T - t) + \beta_1 (T - t) \int_0^T dW_I(u) + \int_t^T a_0(u)L du - b_3(t, T),$$

where L is the constant loss rate, and $B(t, T) \equiv \exp(-y_T(t)(T - t))$ and

$$b_3(t, T) = a_3(1 + a_3)b_2(t, T) + (1/2)\beta_1^2(T - t)^3/3 \\ + a_3\rho\beta_1 \int_t^T b(u, T)(T - u) du.$$

Letting

$$\Delta\chi(t) \equiv \chi(t + 1, T) - \chi(t, T),$$

we obtain

$$\Delta\chi(t) = a_1L[y_{T-1}(t + 1) - y_T(t)] + \beta L \left[\frac{\Delta I(t)}{I(t)} - r(t)\Delta t \right] - c_3(t, T),$$

where

$$c_3(t, T) = [(b_3(t + 1, T)/(T - t - 1)) - (b_3(t, T)/(T - t))].$$

This expression is similar in form to an expression used by Longstaff and Schwartz (1995a, b). It can be used to facilitate estimation of the model's parameters or testing the validity of the model. This addresses one of the concerns raised in the recent Basle Committee on Banking Supervision (1999) report.

4.2. Correlation

The issue of correlation is of central importance in all the credit risk methodologies. Two types of correlation are often identified: default correlation and event correlation. Default correlation refers to firm default probabilities being correlated due to common factors in the economy. For example, default rates increase if the economy goes into a recession (see Altman, 1983/1990; Wilson, 1997a,b). Event correlation refers to how a firm's default probability is affected by default of other firms. This has been modeled by the use of indicator functions and copula functions.²¹

The difficulty with modeling event correlations is that they are, in general, state dependent. For example, consider an industry where one of the major players defaults. Whether this has a positive or negative effect upon the remaining firms depends upon whether default is caused by the economy being in recession or poor management. In the first case, the event correlation may be minimal. In the second case, the event correlation may be significant. The probabilities of default may decrease for the remaining firms because the demise of a competitor allows them to sell more products. Alternatively, if a firm sells the majority of its output to the defaulting firm then this will have a detrimental effect upon the surviving firm.

²¹ Copula functions are described in Bowers et al. (1997). For a different approach see Duffie and Singleton (1998).

In the two factor model described by expression (4.3), default probabilities across obligors are correlated due to their dependence upon common factors. If the coefficients α and β in expression (4.3) are identically zero, then the correlation among default probabilities is zero. This does not imply, however, that the change in the values of credit risky bonds are independent. Their values will be related due to their common dependence upon the underlying term structure of default free interest rates. The effects of correlation must also be considered when estimating the dollar cost of counterparty risk.²² This cost is ignored by most standard pricing models.

4.3. Claims of bond holders

The modeling of the recovery rate process is a crucial component in any credit risk model. A common assumption in the academic literature for the recovery rate, following Duffie and Singleton (1997), is that the value of, say, a zero-coupon bond in default is proportional to its value just prior to default. An alternative assumption often used in industry is based upon the legal claims of bond holders in default. Under this assumption, the value of a zero-coupon bond in default is proportional to the implicit accrued interest. For coupon bonds, the bond holders in default is accrued interest plus face value.

We consider the implication of these two different assumptions for pricing risky zero-coupon and coupon bonds.

4.3.1. Risky zero-coupon bonds

This section values risky zero-coupon bonds under the two different recovery rate assumptions.

Proportional loss. Duffie and Singleton (1997) assume that if default occurs, the value of the zero-coupon bond is

$$v(t, T) = \delta v(t^-, T), \quad (4.8)$$

where $v(t^-, T)$ denotes the value of the bond an instant before default, δ is the recovery rate, and $v(t, T)$ is the value of the bond given default. Following Lando (1994/1997), Duffie and Singleton (1998) and Hughston (1997), the value of a risky zero-coupon bond is given by

$$v_1(t, T) = 1(\Gamma > t)E_t^Q \left[\exp \left(- \int_t^T r(u) + \lambda(u)L(u) du \right) \right], \quad (4.9)$$

where $L(u) \equiv 1 - \delta(u)$ denotes the proportional loss in the event of default.

²² Jarrow and Turnbull (1996a, pp. 577–579), show how to estimate the cost arising from counterparty risk.

*Legal claim approach*²³. An alternative approach consistent with the legal claims of the bond holders assumes that if default occurs, the bond holder’s claim is limited to the implicit accrued interest. Let the bond be issued at time t_0 and its value at the time of issue be denoted by v_0 .

The implicit interest rate, r_i , is defined by

$$v_0 = \frac{1}{1 + r_i(T - t_0)} \tag{4.10a}$$

or

$$r_i = \frac{1}{T - t_0} \left(\frac{1}{v_0} - 1 \right). \tag{4.10b}$$

In the event of default at time Γ , the bond holder’s claim is $v_0[1 + r_i(\Gamma - t_0)]$. The payoff to the zero-coupon bond considering default is

$$\begin{cases} 1 & \text{if } \Gamma > T, \\ \delta v_0[1 + r_i(\Gamma - t_0)] & \text{if } \Gamma \leq T. \end{cases} \tag{4.11}$$

The time t value of the zero-coupon bond is

$$\begin{aligned} v_2(t, T) = E_t^Q \left[\exp \left(- \int_t^T r(s) ds \right) 1(\Gamma > T) \right] \\ + v_0 E_t^Q \left[\exp \left(- \int_t^\Gamma r(s) ds \right) \delta [1 + r_0(\Gamma - t_0)] \right]. \end{aligned} \tag{4.12}$$

Using the results of Lando (1994/1997), as described in Section 2.2, we can write the above expression as

$$\begin{aligned} v_2(t, T) = 1(\Gamma > t) E_t^Q \left[\exp \left(- \int_t^T r(u) + \lambda(u) du \right) \right] \\ + m + 1(\Gamma > t) v_0 E_t^Q \left\{ \int_t^T \delta [1 + r_i(s - t_0)] \lambda(s) \exp \left[- \int_t^s r(u) + \lambda(u) du \right] ds \right\}. \end{aligned} \tag{4.13}$$

The recovery rate process determines the form of the zero-coupon bond price. This is important for both pricing and estimation. If the recovery rate is given by expression (4.8), then expression (4.9) describes the bond price. If the recovery rate is given by expression (4.11), then expression (4.13) describes the bond price.

²³ The legal claims approach is used by a number of practitioners. This section simply collects together what seems to be common ‘street’ knowledge.

4.3.2. Credit risky coupon bonds

This section values a coupon bond under the two different assumptions about the recovery rate process in the event of default. Consider a risky bond that promises to make coupon payments $\{c_j\}$ at time $\{t_j\}$, $j = 1, \dots, n$ where n is the number of promised payments. The principal, F , is paid at time t_n . Let $v_c(t)$ denote the time t value of the bond, conditional upon no default.

Proportional loss. Using expression (3.2) gives

$$\begin{aligned}
 v_{1c}(t) &= E_t^Q \left\{ \sum_{j=1}^n c_j \exp \left(- \int_t^{t_j} r(u) + \lambda(u)L(u) du \right) \right. \\
 &\quad \left. + F \exp \left(- \int_t^{t_n} r(u) + \lambda(u)L(u) du \right) \right\} \tag{4.14} \\
 &= \sum_{j=1}^n c_j v_1(t, t_j) + F v_1(t, t_n).
 \end{aligned}$$

The usual value additivity result holds.

Legal claim approach. In the event of default, the bond holders' claim is limited to accrued interest plus principal. The implicit legal assumption is that bonds are trading at par value.

If default occurs over the first period, the payoff is

$$\delta [Fr_i(\Gamma - t_0) + F] \quad \text{for } t < \Gamma \leq t_1,$$

where t_0 is the time of the last coupon payment, and r_i the coupon rate. The first term inside the square brackets is the accrued interest and the second term is the principal.

Conditional upon no default prior to time t_{j-1} , if default occurs over the period (t_{j-1}, t_j) , the payment to bond holders is

$$f_j(\Gamma) \equiv \delta [Fr_i(\Gamma - t_{j-1}) + F] \quad \text{for } t_{j-1} < \Gamma \leq t_j \text{ where } j=1, \dots, n. \tag{4.15}$$

The value of this claim at time t_{j-1} is

$$\begin{aligned}
 v_j(t_{j-1}) &= E_{t_{j-1}}^Q \left[\exp \left(- \int_{t_{j-1}}^{\Gamma} r(u) du \right) f_j(\Gamma) \right] \\
 &= 1(\Gamma > t_{j-1}) E_{t_{j-1}}^Q \left[\int_{t_{j-1}}^{t_j} f_j(s) \lambda(s) \exp \left(- \int_{t_{j-1}}^s r(u) + \lambda(u) du \right) ds \right]. \tag{4.16}
 \end{aligned}$$

The value of this claim at time t is

$$\begin{aligned}
 v_j(t) &= E_t^Q \left[\exp \left(- \int_t^{t_{j-1}} r(x) dx \right) v_j(t_{j-1}) 1(\Gamma > t_{j-1}) \right] \\
 &= 1(\Gamma > t) E_t^Q \left[\exp \left(- \int_t^{t_{j-1}} r(x) + \lambda(x) dx \right) v_j(t_{j-1}) \right] \\
 &= 1(\Gamma > t) E_t^Q \left\{ \exp \left(- \int_t^{t_{j-1}} r(x) + \lambda(x) dx \right) \right. \\
 &\quad \left. \int_{t_{j-1}}^{t_j} f_j(s) \lambda(s) \exp \left(- \int_{t_{j-1}}^s r(u) + \lambda(u) du \right) ds \right\} \\
 &= 1(\Gamma > t) E_t^Q \left[\int_{t_{j-1}}^{t_j} f_j(s) \lambda(s) \exp \left(- \int_t^s r(u) + \lambda(u) du \right) ds \right]. \tag{4.17}
 \end{aligned}$$

Using the above result, the value of the coupon bond is given by

$$\begin{aligned}
 v_{2c}(t) &= E_t^Q \left\{ \left[\sum_{j=1}^n c_j \exp \left(- \int_t^{t_j} r(u) du \right) \right. \right. \\
 &\quad \left. \left. + F \exp \left(- \int_t^{t_n} r(u) du \right) \right] 1(\Gamma > t_n) \right\} + \sum_{j=1}^n v_j(t) \\
 &= 1(\Gamma > t) E_t^Q \left[\sum_{j=1}^n c_j \exp \left(- \int_{t_{j-1}}^{t_j} r(u) + \lambda(u) du \right) \right. \\
 &\quad \left. + F \exp \left(- \int_t^{t_n} r(u) + \lambda(u) du \right) \right] \\
 &\quad + 1(\Gamma > t) E_t^Q \left[\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \delta F [1 + r_i(s - t_{j-1})] \lambda(s) \right. \\
 &\quad \left. \exp \left(- \int_t^s r(u) + \lambda(u) du \right) ds \right]. \tag{4.18}
 \end{aligned}$$

This result is additive, but not the form of expression (4.14). This implies that the standard stripping procedures used to determine the implied zero-coupon bonds do not apply.

4.4. Convenience yields on treasury securities

In the Jarrow–Turnbull model the credit spread is used to infer the default probability under the equivalent martingale measure. Many factors, such as restrictions on short selling, illiquidities, regulatory requirements and taxation,

may affect the spread. Babbs (1991) and Grinblatt (1994) argue that a convenience yield partly explains the spread between the Euro-dollar and Treasury term structure. This convenience yield is an implication of short sale constraints on Treasury securities that occasionally exist – see Cornell and Shapiro (1986), Duffie (1996), and Chatterjea and Jarrow (1998). Following Jarrow and Turnbull (1997b), we show how to augment the Jarrow–Turnbull model to include a convenience yield.

Let $b(t, T)$ denote the time t price of a non-shortable zero-coupon Treasury bond that matures a time T .²⁴ The no-arbitrage relationship between $b(t, T)$ and a zero-coupon Treasury bond not subject to short sell restrictions is

$$b(t, T) \geq B(t, T). \tag{4.19}$$

A strict inequality is possible if the short selling constraint is binding.

Let $y(t, T)$ denote the forward convenience yield. Using the forward convenience yield, the above expression can be written as

$$b(t, T) \exp\left(-\int_t^T y(t, s) ds\right) = B(t, T), \tag{4.20}$$

where $y(t, s) \geq 0$ for all $0 \leq t \leq s \leq T$.

Recall that no arbitrage between $B(t, T)$ and $v(t, T)$ implies that $B(t, T)/A(t)$ and $v(t, T)/A(t)$ are Q -martingales. Using expression (4.20), this implies that

$$b(t, T) \exp\left(-\int_t^T y(t, s) ds\right) / A(t) \tag{4.21}$$

is a Q martingale. Given exogenous specifications for $B(t, T)$ and $y(t, T)$ under the measure Q , expression (4.21) determines the arbitrage free stochastic process for $b(t, T)$. $B(t, T)$ can be modeled using standard techniques (see Heath et al., 1992). To model $y(t, T)$, we rewrite expression (4.21).

The spread between a credit risky zero-coupon bond and a zero-coupon non-shortable Treasury bond is

$$v(t, T)/b(t, T) = [v(t, T)/B(t, T)] \exp\left[-\int_t^T y(t, s) ds\right]. \tag{4.22}$$

Define

$$Y(t, T) \equiv \exp\left[-\int_t^T y(t, s) ds\right]. \tag{4.23}$$

$Y(t, T)$ has the properties of a zero-coupon bond and $y(t, T)$ the properties of a non-negative forward rate. Consequently, $\{y(t, T)\}$ can also be modeled along the lines described in Heath et al. (1992).

²⁴ The term “non-shortable” refers to a case where there are restrictions on the amount of securities that can be shorted.

4.5. Change of probability measure

In credit risk management, it is necessary to keep track of two probability measures: the natural measure and the equivalent martingale measure. While this adds an extra layer of complexity, it also generates some interesting benefits.

From the use of bond spreads, we can infer the probability of no default over a specified horizon T under the probability measure Q :

$$\Pr^Q[\Gamma > T] = E_t^Q \left[\exp \left(- \int_0^T \lambda(u) du \right) \right]. \quad (4.24)$$

If we can estimate the price of risk for the underlying factors, we can change the probability measure and estimate the probability of no default under the natural measure

$$\Pr^P[\Gamma > T] = E_t^P \left[\exp \left(- \int_0^T \lambda(u) du \right) \right]. \quad (4.25)$$

This has an important practical implication. It provides a method to check the estimates of default probabilities generated either internally, by credit rating agencies, or by other commercial packages such as KMV.

Some forms of credit derivatives are contingent upon credit events, such as credit rating downgrades. To price such instruments requires a model that explicitly incorporates credit rating changes. The pricing of such derivatives is usually done using the equivalent martingale or risk-neutral probabilities. Jarrow et al. (1997) show how to incorporate credit ratings into the arbitrage free pricing of credit risky derivatives. They show how to infer the martingale transition probabilities given the transition probabilities under the natural measure. The Jarrow–Lando–Turnbull model has been extended by Das and Tufano (1996) and Monkkonen (1997). Das and Tufano assume that in the event of default the recovery rate is a random variable correlated with the default free rate of interest. The independence assumption between the transition probabilities and the default free rate of interest is maintained. Monkkonen generalizes Das and Tufano by allowing the probabilities of default to depend upon the default free rate of interest. The work of Monkkonen can be generalized further by modeling the transition probabilities as Cox processes (see Lando, 1994/1997). The only difficulty is that of estimating the transition matrix coefficients.

5. Summary

Economic theory tells us that market and credit risk are related to each other and not separable. This lack of separability affects the determination of

economic capital. It affects the risk adjusted return on capital used in measuring the performance of different groups within a bank, and it affects the calculation of the value-at-risk, all of which are important to regulators.

With accrual accounting the only risk associated with a loan is default. Current methodologies such as CreditMetrics, CreditRisk+, and KMV emphasize the accrual accounting perspective and focus on only default risk. Interest rates are assumed to be constant, implying that these methodologies cannot assess the risk associated with interest rate derivatives. In contrast, reduced form models, such as the Jarrow–Turnbull model, consider market and credit risk. They can be calibrated using observable data and consequently incorporate market information. They can be used for pricing and for risk management.

6. For further reading

The following references are also of interest to the reader: Altman (1968, 1987, 1989, 1993, 1996); Altman and Kao (1992a,b); Wei (1995); Weiss (1990); Altman and Nammacher (1985); Amin and Jarrow (1991, 1992); Anderson and Sundaresan (1996); Asquith et al. (1994, 1989); Barclay and Smith (1995); Basle Committee (1996); Bensoussan et al. (1994); Black and Cox (1976); Black et al. (1990); Blume et al. (1991); Cathcart and El-Jahel (1998); Chance (1990); Cooper and Mello (1990a,b); Cornell and Shapiro (1989); Cornell and Mello (1991) Credit Matrics (1997); Crouhy and Galai (1997); Delienedis and Geske (1998); Duan (1994); Duffee (1997, 1999); Duffie and Huang (1996); Duffie and Singleton (1994a,b); Eberhart et al. (1990); Flesaker et al. (1994); Harrison and Pliska (1981); Helwege and Kleiman (1997); Ho and Lee (1986); Ho and Singer (1982, 1984); Hull and White (1996); Jacod and Shiryaev (1987); Jarrow and Madan (1995); Jarrow and Turnbull (1994, 1998); Altman and Bencivenga (1995); Johnson and Stulz (1987); Kijima (1998); Kim et al. (1993); Lando (1997, 1998); Leibowitz et al. (1995); Li (1998); Altman and Eberhart (1994); Madan and Unal (1994); Merton (1976); Musiela et al. (1993); Schonbucher (1998); Schwartz (1993, 1997, 1998); Shimko et al. (1993); Singleton (1997); Skinner (1994); Titman and Torous (1989); Wakeman (1996).

Acknowledgements

We thank the Editor and two anonymous referees and seminar participants at the “CreditRisk Modelling and the Regulatory Implications” conference organized by the Bank of England and the Financial Services Authority, the Bank of Japan, the Federal Reserve Board of the United States, and the

Federal Bank of New York; Rotman School of Management, University of Toronto; Columbia University; the Field’s Institute; the Federal Reserve Bank of New York; and the Central Bank of Sweden for numerous comments and suggestions.

Appendix A. Two factor model

The value of a zero-coupon credit risky bond is

$$v(t, T, \bar{D}) = \exp\left(-\int_t^T a_0(u) du\right) E_t^Q \left[\exp\left(-a_2 \int_t^T r(u) + \beta_1 W_I(u) du\right) \right], \tag{A.1}$$

where $a_2 \equiv 1 + a_1L$, $a_3 \equiv a_1L$, and $\beta_1 \equiv \beta\sigma_1L$. Now consider

$$a_2 \int_t^T r(u) du = a_2 \left\{ \int_t^T f(0, u) du + (\sigma^2/2) \int_t^T b(0, u)^2 du + b(t, T)\sigma \int_0^t \exp[-\vartheta(t-s)] dW(s) + \sigma \int_t^T b(s, T) dW(s) \right\} \tag{A.2}$$

and

$$\begin{aligned} \beta_1 \int_t^T W_I(u) du &= \beta_1 \int_t^T du \int_0^u dW_I(u) \\ &= \beta_1 \left[(T-t) \int_0^t dW_I(u) + \int_t^T (T-s) dW_I(s) \right]. \end{aligned} \tag{A.3}$$

Hence

$$\begin{aligned} E_t^Q \exp \left[-a_2\sigma \int_t^T b(s, T) dW(s) - \beta_1 \int_t^T (T-s) dW_I(s) \right] \\ = \exp \left[\frac{1}{2} a_2^2 \sigma^2 \int_t^T b(s, T)^2 ds + \frac{1}{2} \beta_1^2 \int_t^T (T-s)^2 ds + a_2\sigma\beta_1 \int_t^T b(s, T)(T-s)\rho ds \right], \end{aligned} \tag{A.4}$$

where ρ is the correlation coefficient. The value of a zero-coupon credit risky bond is given by

$$v(t, T, \bar{D}) = B(t, T) \exp \{ -a_3 b(t, T) [r(t) - f(0, t)] + c_2(t, T) - \beta_1 (T - t) W_I(t) \}. \quad (\text{A.5})$$

Let $\chi(t, T)$ denote the credit spread, then

$$v(t, T) = 1(\Gamma > t) B(t, T) \exp[-\chi(t, T)(T - t)]$$

implying

$$\chi(t, T)(T - t) = a_3 b(t, T) [r(t) - f(0, t)] + \beta_1 (T - t) W_I(t) - c_2(t, T).$$

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