Bank of America and CCAR 2016 Stress Testing:  
A Simple Model Validation Example  
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The author wishes to thank his colleague, Managing Director for Research Prof. Robert A. Jarrow, for twenty years of guidance and helpful conversations on this critical topic.

During the recent financial crisis, a large number of financial services firms failed and a still larger number of them would have failed without government assistance. The Kamakura Public Firm Default Probabilities Technical Guide (Jarrow et al, June, 2015) for the Kamakura Risk Information Services default probability service shows that the number of failed financial services firms in the recent credit crisis was very significant:
The Dodd-Frank Act stress tests and the related Federal Reserve Comprehensive Capital Analysis and Review program were implemented in order to more accurately assess the default risk of financial institutions on a forward-looking basis. This note shows the best practice for stress testing the default probabilities of public firms from two perspectives: as an insider using the full knowledge of the firm’s assets and liabilities, and as an outsider, using only publicly available information about the firm. We show that a commonly used approach to stress testing fails normal model validation procedures from both model quality and econometric procedural points of view. We then show that the use of the reduced form approach to credit modeling is both more accurate and more transparent than the common practice approach.

Credit Spreads and Default Probabilities in the Credit Crisis

The dramatic events of the recent credit crisis can be seen clearly in this graph provided by Kamakura Corporation of the credit spreads on the Bank of America Corporation (BAC) 5.125% bonds due November 15, 2014. At the time of Lehman Brothers bankruptcy filing on September 15, 2008, the bonds had a maturity of slightly more than six years. Credit spreads on the bonds after the bankruptcy of Lehman Brothers ballooned out to nearly 1000 basis points, even with the publicly stated support of the Federal Reserve and the U.S. Treasury.
Credit spreads on the 8.5% Lehman Brothers Holdings Inc. (LEHMQ) bond issue due August 1, 2015 show the explosion in credit spreads as the September 15, 2008 bankruptcy filing date approaches. Credit spreads after the bankruptcy filing reflect the implied recovery by bond holders.

Kamakura Corporation KRIS version 6.0 Jarrow-Chava reduced form 1 year default probabilities for both Bank of America Corporation (in yellow) and Lehman Brothers (in blue) are shown below:
The stress of the credit crisis is apparent both in terms of default probabilities and credit spreads. The objective of both the Dodd-Frank Act stress tests ("DFAST") and the Federal Reserve CCAR program is a simulation of both default probabilities and credit spreads in an environment similar to the 2008-2010 period at the heart of the credit crisis. We now turn to a summary of common practice and best practice approaches to stress testing.

Common Practice and Best Practice Default Probability Stress Tests

Both default probabilities and credit spreads need to be stress tested to fulfil the requirements of the DFAST and CCAR programs. For “own risk self-assessment” ("ORSA") and counterparty credit risk, the data environment varies as summarized in this graphic:

<table>
<thead>
<tr>
<th>Time Horizon of Stress Testing</th>
<th>Subject of Stress Testing Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current and Historical Default Probabilities</td>
<td>Own Risk Self-Assessment</td>
</tr>
<tr>
<td>Assets and Liabilities</td>
<td>All known with certainty.</td>
</tr>
<tr>
<td>Stock price and recent performance</td>
<td>Known with certainty.</td>
</tr>
<tr>
<td>Credit spreads and recent history</td>
<td>Known with certainty.</td>
</tr>
<tr>
<td>Macro factors</td>
<td>Known with certainty.</td>
</tr>
<tr>
<td>Forward Looking Default Probabilities</td>
<td></td>
</tr>
<tr>
<td>Assets and Liabilities</td>
<td>Partially known.</td>
</tr>
<tr>
<td>Stock price and recent performance</td>
<td>Unknown.</td>
</tr>
<tr>
<td>Credit spreads and recent history</td>
<td>Unknown.</td>
</tr>
<tr>
<td>Macro factors</td>
<td>Assumed to be known.</td>
</tr>
</tbody>
</table>

Stress testing involves both own risk self-assessment and counterparty stress testing. The table above makes it clear that the only inputs that are “known” inputs for the calculation of future default probabilities are the macro-economic factors required to be used in the stress test. The word “known” in the prior sentence is in fact “assumed to be known.” The general steps in stress testing are as follows:

- Using all available public and internal information, calculate the current and historical default probabilities for one’s own institution and all counterparties for as long a history as possible using a granular time interval (monthly is best
practice). In the case of the Kamakura KRIS default probability service, default probabilities are available from January 1990 to the present.

- Fit a mathematical function linking the default probability of each legal entity (for both one’s own institution and for every counterparty) to the set of macro-economic factors which will be used for stress testing plus any additional macro factors that bring added insight or explanatory power.
- Use these mathematical functions to calculate the change in default probabilities for all entities at each time horizon specified in the stress test for all scenarios required in the stress test (a Monte Carlo simulation is necessary to validate the stress test assumptions).
- Simulate default/no default given the simulated default probabilities and summarize all relevant market valuation and financial accounting information.

Common practice and best practice differ at each step. Many different approaches have been used in stress tests around the world. We summarize the most common approaches here.

1. Simulation based on lagged default probabilities
2. Merton-based stress testing
3. Reduced form stress testing using macro factors as the only input
4. Reduced form stress testing using macro factors and lagged company specific inputs

We turn to a discussion of each method using the example of Bank of America Corporation for a counter-party credit risk model validation perspective.

**Default Probability History Used**

The next graph shows the history of three month default probabilities using both the KRIS version 6.0 Merton default probability (in yellow) and the KRIS version 6.0 Jarrow-Chava reduced form default probability (in blue) for Bank of America Corporation:

Plotting the same data on a log scale shows that the differences in the default models are considerable over the 1990 to 2015 time period:
Method 1: Simulation based on lagged default probabilities

Simulation of forward-looking default probabilities based on a function of their lagged value is a common starting point for firms early in the stress-testing process when data and time is limited. We illustrate the problems that emerge from this technique in our worked example below.

Method 2: Merton-based stress testing

For a general introduction to the differences between the Merton (1974) model and reduced form credit models, Hilscher, Jarrow and van Deventer (2008) present a summary of how both types of models work. It is well-known that, despite the intuitive appeal of the model, the Merton model has been disappointing from an accuracy perspective. Campbell, Hilscher and Szilagyi (2008) demonstrated that the reduced form approach to default modeling was substantially more accurate than the Merton model of risky debt. Bharath and Shumway (2008), working completely independently, reached the same conclusions. A follow-on paper by Campbell, Hilscher and Szilagyi (2011) confirmed their earlier conclusions in a paper that was awarded the Markowitz Prize for best paper in the *Journal of Investment Management* by a judging panel that included Prof. Robert Merton. Kamakura Corporation, working independently of Bharath, Shumway, Campbell, Hilscher and Szilagyi, also showed in five Kamakura Risk Information Services Technical Guides (May 2003, December 2003, January 2006, March 2011, and June, 2015) that the Merton model of risky debt is less accurate than the reduced form approach at all time horizons. Jarrow and van Deventer (1998, 1999, reprinted 2004) were among the first to show the Merton model was both inconsistent with actual movements of stock prices and credit spreads and that the econometric foundation for estimating default probabilities is very weak. A portion of Jarrow and van Deventer (1999, reprinted 2004) is reproduced in Appendix A. Campbell, Hilscher, and Szilagyi (2008, page 2915) explain the reason that the Merton model fails the “effective challenge” model validation test very clearly:

“If one’s goal is to predict failures, however, it is clearly better to use a reduced-form econometric approach that allows volatility and leverage to enter with free coefficients and that includes other relevant variables. Bharath and Shumway (2008), in independent recent work, reach a similar conclusion.”
There is another very important model validation problem with Merton-based default probability estimation and fitting to historical macro-economic factors. Appendix A from Jarrow and van Deventer (1999, reprinted 2004) shows that the default probability formula in the Merton model takes this form for company $i$, where $B$ is the value of debt which must be repaid at the end of single period, $V$ is the value of company assets, $\mu$ is the drift in the return on company assets, $\sigma$ is the volatility of the return on company assets, and $k$ is the ratio of (the drift in the return on the total assets of all companies) to the risk free rate $r$:

$$\Pr(\text{Default}) = \Pr(V_i(t) < B) = N\left(\frac{\ln(B/V_i(0)) - \mu t}{\sigma \sqrt{t}}\right)$$

$$\mu_i = -\frac{1}{2} \sigma_i^2 + (1 - b_i + b_i k) r.$$

The common practice approach to fitting the Merton approach to macro factors takes these steps:

- The unobservable value of company assets and the unobservable volatility of company assets are implied from historical stock price movements. See van Deventer, Imai and Mesler (2013, appendix to Chapter 18) for the derivation of the implied values of $V$ and $\sigma$.
- An index of the value of assets for all companies in the same industry and geographical region is constructed.
- A regression is run which links the return on the assets of all companies in that “industry-region” with the return on the assets of company $i$. The error term of the regression is “the idiosyncratic component” of credit risk for company $i$.
- After the fact, the variation of the “industry region” asset returns is in turn fit to a specified set of macro factors to allow factor-specific stress testing of Merton default probabilities.

This four stage procedure and other multi-stage procedures were discussed at length by Angrist and Pischke (2009). Van Deventer (2014) relates their conclusions in detail:

“Angrist and Pischke describe the consequences of this modeling choice. Angrist is a professor of economics at MIT and was listed by Thomson Reuters as being on the short list for the Nobel Prize in 2013. Pischke is a professor at the London School of Economics. Here is their comment on the validity of this modeling strategy from Angrist and Pischke (2009, p. 190):

“Forbidden regressions were forbidden by MIT professor Jerry Hausman in 1975, and while they occasionally resurface in an under-supervised thesis, they are still technically off-limits. A forbidden regression crops
up when researchers apply 2SLS [two stage least squares] reasoning directly to non-linear models.”

“They go on to say (2009, page 192)

“As a rule, naively plugging in first-stage fitted values in non-linear models is a bad idea. This includes models with a non-linear second stage as well as those where the [conditional expectation function] for the first stage is non-linear. “

“Earlier (page 122), Angrist and Pischke address the simpler case where both the first stage and the second stage are linear:

“The [two stage least squares] name notwithstanding, we don’t usually construct 2SLS estimates in two steps. For one thing, the resulting standard errors are wrong...Typically, we let specialized software routines...do the calculation for us. This gets the standard errors right and helps to avoid other mistakes.”

“We refer interested readers to Section 4.6 of Angrist and Pischke (2009) for the full details of the problems with this pseudo-two stage approach to generating 13 quarter scenarios for the default probabilities of ABC Company.”

The common practice Merton approach to stress testing is even worse that the case discussed by Angrist and Pischke, because it has 4 stages, not 2.

A final model validation issue with the Merton approach is the use of Gaussian copulas to simulate the portfolio losses using the stress-tested default probabilities. It is well-known (and in fact sometimes forgotten) that the Gaussian copula was widely discredited because it grossly understated the risk of collateralized debt obligations in the 2006 to 2010 credit crisis. The most famous critiques appeared on page one of the Wall Street Journal (September 12, 2005) and in a Felix Salmon piece for Wired magazine with this provocative title:
We conclude that the Merton approach to stress testing fails normal validation procedures for three important reasons:

- Default probability estimates developed from Merton procedures fail the “effective challenge” test, because reduced form models produce more accurate results. Stress test analysis built on Merton default probabilities is a house built on sand.
- The four stage econometric procedure used to fit these default probabilities to macro factors is a “forbidden model” as explained by Angrist and Pischke.
Inappropriate econometric procedures alone are grounds for a model validation rejection.

- Portfolio simulation using the Gaussian copula methodology (still distributed by a rating agency) is widely recognized as inaccurate. The critiques are widely noted in the popular press and it is perfectly appropriate to attribute some of the severity of the last credit crisis to the model risk of the Gaussian copula.

For these reasons, we ignore the Merton model in what follows and compare methods 1, 3, and 4.

**A Worked Example Using Reduced Form Default Probabilities**

In this section, we use the Kamakura Risk Information Services version 6.0 Jarrow-Chava reduced form default probability model (abbreviated KDP-jc6), which makes default predictions using a sophisticated combination of financial ratios, stock price history, and macro-economic factors. The version 6.0 model was estimated over the period from January 1990 to May 2014, and includes the insights of the recent credit crisis. Kamakura default probabilities are based on 2.2 million observations and more than 2,700 defaults. The term structure of default is constructed by using a related series of econometric relationships estimated on this data base. KRIS covers 36,000 firms in 61 countries, updated daily. An overview of the full suite of Kamakura default probability models is available [here](#).

Our hypothetical counterparty is Bank of America Corporation.

We compare the accuracy of three methods of linking historical default probabilities to macro factors and other explanatory variables:

- Method 1, where we predict future default probabilities from historical default probabilities.
- Method 3, where we predict future default probabilities as a function of “assumed known” macro factors alone
- Method 4, where we predict future default probabilities as a function of “assumed known” macro factors and lagged historical company attributes.

We can, in theory, fit the macro factors either to (a) a time series of actual defaults (default flag = 1) or non-defaults (default flag = 0) or to (b) a time series of historical default probabilities that are assumed to be true. Because we are doing a company specific (Bank of America) model, method (b) is the only choice we have.

Best practice econometric procedures dictate how the estimation is done:

- The link between default probabilities and macro factors is done in one stage, since the formulas used are non-linear and we want to avoid the Angrist and Pischke “forbidden model” problem.
- The non-linear function selected to produce the default probabilities should be such that all forecast default probabilities fall in the range from 0 to 1 (i.e. 100%).

The two most common choices for a default probability function is a cumulative probability function like that of the normal or logistic distribution. Chava and Jarrow
have proven that the logistic function is the maximum likelihood estimator for the default/no default 0/1 dependent variable case. For purposes of today’s example, we use the logistic function to predict the default probability at time t based on n functions of macro factors Xᵢ for i = 1,n:

\[ P[t] = \frac{1}{1 + e^{-\alpha - \sum_{i=1}^{n} \beta_i X_i[t]}} \]

Econometrically, we can use any one of a number of methods to fit the parameters of this formula after we specify the functions Xᵢ of the m macro factors that we will use as candidate variables. Some common choices are:

- General linear methods
- Maximum likelihood estimation
- Non-linear least squares estimation
- Fractional regression

There is another alternative. If we transform both sides of equation, we can explain a transformation Z of the default probabilities as a linear function of our explanatory variables, which allows us to use ordinary least squares:

\[ Z[t] = -\ln\left( \frac{1 - P[t]}{P[t]} \right) = \alpha + \sum_{i=1}^{n} \beta_i X_i \]

For maximum transparency, we make this transformation and use ordinary least squares in what follows. As a practical matter, the differences in coefficients among the methods have been small in our experience.

**Regression Estimation Strategy**

We now need to fit one relationship for each period for which we need a forecasted default probability at quarter k given macro factors at time k and our company specific inputs at time 0. Consider the case where the set of explanatory variables includes the lagged default probability (which is only “known” at time 0 and historical dates) and lagged financial ratios, say the net income to assets ratio. We also assume that there are n functions of the 28 CCAR macro factors. One such system of equations that generates the relevant 13 quarters of forecasts as a function of our time zero inputs and the 28 macro factors Xᵢ specified in CCAR is given here. The macro factors are not lagged unless the econometric process reveals that lags are helpful. Negative numbers in parentheses denote lagged values, which effectively cause the time 0 values to be used as explanatory variables:

Quarter 1: PD=f[PD(-1),NI/A(-1), R(-1), X₁, X₂,….Xₙ]
Quarter 2: PD=f[PD(-2),NI/A(-2), R(-2), X₁, X₂,…Xₙ]
Quarter 3: PD=f[PD(-3), NI/A(-3), R(-3), X1, X2, ... Xn]
Quarter 4: PD=f[PD(-4), NI/A(-4), R(-4), X1, X2, ... Xn]
Quarter 5: PD=f[PD(-5), NI/A(-5), R(-5), X1, X2, ... Xn]
Quarter 6: PD=f[PD(-6), NI/A(-6), R(-6), X1, X2, ... Xn]
Quarter 7: PD=f[PD(-7), NI/A(-7), R(-7), X1, X2, ... Xn]
Quarter 8: PD=f[PD(-8), NI/A(-8), R(-8), X1, X2, ... Xn]
Quarter 9: PD=f[PD(-9), NI/A(-9), R(-9), X1, X2, ... Xn]
Quarter 10: PD=f[PD(-10), NI/A(-10), R(-10), X1, X2, ... Xn]
Quarter 11: PD=f[PD(-11), NI/A(-11), R(-11), X1, X2, ... Xn]
Quarter 12: PD=f[PD(-12), NI/A(-12), R(-12), X1, X2, ... Xn]
Quarter 13: PD=f[PD(-13), NI/A(-13), R(-13), X1, X2, ... Xn]

Normally, in practice, as the time horizon lengthens it becomes more and more likely that the time zero inputs lose their statistical significance and are dropped from the model. For models 1, 3 and 4, we use the following procedures:

Method 1 uses the lagged 3 month default probability so we have a nesting of equations like these above.

Method 3 uses macro factors as the only explanatory variables, so a simple implementation is a single regression. We have no lagged explanatory variables in this simple implementation.

Method 4 uses the macro factors as explanatory variables plus we allow financial ratios and other company specific inputs known at time zero to be candidate variables. Because we have these lagged variables, we need 13 equations.

Results of the Model Fitting for Bank of America Corporation

The CCAR macro factors were released so early in 2016 that it was impossible to know the true values of some of the 28 CCAR macro factors. For that reason, we actually project out of sample for 14 quarters, not 13 quarters, using the third quarter of 2015 as our time zero date. We project through the first quarter of 2019.

We start with the r-squared for the three models on the transformed variable Z, the logit of the unannualized default probability expressed as a decimal. Here are the results:
Method 1 is graphed in red. Using only the time zero value of the default probability has very little explanatory power, approaching zero r-squared as we get to lags as long as 14 quarters. We need only one regression for Method 3 (in orange), in which we use only the functions of macro factors as candidate variables. There are no lags, so there is just one regression and the r-squared is the same for all of the 14 quarterly forecast horizons. The green line shows the r-squareds for the 14 regressions of method 4, in which we allow the time zero company specific candidate variables to enter with the appropriate lag. In the case of Bank of America Corporation, this method did not perform as well as Method 3. The results are mixed for other CCAR banks.

Next we look at the root mean squared errors of three methods. They are displayed in this graph. Note that this is the root mean squared error of the transformed variable $Z$, not the raw annualized default probability expressed as a percent:
We now look at the effective r-squared after we transform our prediction of Z back to an annualized default probability, expressed as a percent. The results are given here. Because Method 1 was so clearly inaccurate, we did not display it on the graph:

As one might expect, using time zero company specific inputs does boost short term accuracy but it makes little or no difference in accuracy thereafter.

**Examples of Regression Results**

We now give simple examples of all three methods where we are projecting one quarter ahead, given our assumptions, which is the 4th quarter of 2015.
Method 1 uses only the lagged value of the transformed variable Z. The adjusted r-squared is 23.25% and the lagged value of Z is statistically significant.

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>Number of obs = 102</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>81.3116286</td>
<td>1</td>
<td>81.3116286</td>
<td>F(1, 100) = 31.60</td>
</tr>
<tr>
<td>Residual</td>
<td>257.343261</td>
<td>100</td>
<td>2.57343261</td>
<td>R-squared = 0.2401</td>
</tr>
<tr>
<td>Total</td>
<td>338.65489</td>
<td>101</td>
<td>3.35501871</td>
<td>Adj R-squared = 0.2325</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Root MSE = 1.6042</td>
</tr>
</tbody>
</table>

| lagZKDP3mo | Coef    | Std. Err. | t     | P>|t|     | [95% Conf. Interval] |
|------------|---------|-----------|-------|--------|---------------------|
| _cons      | -4.100065 | .7336363 | -5.59 | 0.000  | -5.555579 -2.644552 |

Method 3 uses only the macro factors and their transformations as explanatory variables. We have 90 observations, down from 102 in the prior equation, because some of the CCAR variables are not available for the full period for which we have default probabilities. We can explain 51.17% of the variation in the transformed variable Z. The statistically significant macro variables are the VIX volatility index, the Japan-U.S. dollar exchange rate, and the 1 year change in the U.S. unemployment rate. We allowed the yen exchange rate to enter as an explanatory variable only for very large international firms like Bank of America and suppressed it as a candidate variable for purely domestic banking institutions.

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>Number of obs = 90</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>155.412801</td>
<td>3</td>
<td>51.8042669</td>
<td>F(3, 86) = 32.09</td>
</tr>
<tr>
<td>Residual</td>
<td>138.823392</td>
<td>86</td>
<td>1.61422548</td>
<td>R-squared = 0.5282</td>
</tr>
<tr>
<td>Total</td>
<td>294.236192</td>
<td>89</td>
<td>3.30602463</td>
<td>Adj R-squared = 0.5117</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Root MSE = 1.2705</td>
</tr>
</tbody>
</table>

| vix     | .0852987 | .0139236 | 6.13  | 0.000  | .0576196 .1129778 |
| jpyusd  | -0.0337915 | .009254 | -3.65 | 0.000  | -.0521878 -.0153951 |
| d1yurusunemp | .3882577 | .1535998 | 2.53  | 0.013  | .0829114 .693604 |
| _cons   | -6.677607 | 1.078037 | -6.19 | 0.000  | -8.820673 -4.53454 |

Method 4 has time zero company specific factors as additional candidate variables. Projecting one quarter forward, we use a one quarter lag on the candidate company specific inputs. We get a sparse relationship that explains the variation in the transformed Z variable for Bank of America as a function of the VIX and the one quarter lagged value of the percentile rank of Bank of America’s common stock price compared to all other common stocks traded in the United States. The adjusted r-squared is 48.56.
Actual Versus Estimated Bank of America Default Probabilities

For the first quarter projections, we get this comparison of the 3 methods versus actual in-sample default probabilities using absolute annualized default probability levels:

Note that Method 3, using macro factors only, dramatically over predicts the default probability at the heart of the credit crisis. The method 1 projection at that time, using only the lagged default probability as an input, dramatically under predicts. The same graph shown on a log scale is given here:
Now we look to the end of the forecast period, 14 quarters forward. The absolute level of in-sample default probabilities versus the forecasts from the 3 models is given here:

Visually, method 4 using time zero company specific inputs, even 14 quarters out, results in a more realistic projection at the peak of the credit crisis. The same graph on a log scale, is shown here:
Stress Test Results

In this section, we shown the stress test response of the estimated Bank of America default probabilities for Methods 3 and 4 over the out-of-sample 14 quarter forecasting horizon. We do not use Method 1 because it has no macro factors as direct inputs. We use the Federal Reserve’s base line, adverse, and severely adverse scenarios. The results for Method 3 (using only macro factors) are shown here:

In the severely adverse scenario using Method 3, the stressed default probability (in red) is near 10%, a much less severe result than actual Bank of America experience.
in the credit crisis. We can display the same stress test on a log scale. That outcome is shown here:

We now turn to Method 4, where time zero company specific explanatory variables are allowed. Looking at absolute levels of the stressed default probabilities, we see that the stressed default probabilities respond in a more muted way than when only macro factors are used as inputs.

The same graph on a log scale is shown here:
Again the results are intuitively appealing and more muted than the model without time zero company specific explanatory variables.

Conclusions

We use a challenger model basis for model validation and examine four common methodologies for conducting Fed CCAR stress tests. We conclude that the Merton model methodology fails on three grounds:

- Failure to exceed accuracy of an “effective challenger” model (the reduced form approach)
- Failure to use proper econometric procedures
- Failure to produce accurate loss estimates, based on the credit crisis experience with the Gaussian copula model.

Method 1 tests were conducted but we found two critical model validation issues: a lack of in-sample accuracy beyond the shortest time horizons and lack of a clear and transparent link to the 28 macro factors specified in the Federal Reserve 2016 CCAR scenarios.

Method 3 (using functions of macro factors as the only explanatory variables) and Method 4 (adding time zero company specific explanatory variables to the macro factors) produce similar results. A comparison over a broad range of counterparties is the appropriate way to distinguish between these two models.

Finally, we note that econometric analysis of a single firm embeds the implicit assumption that the estimated coefficients have remained constant over the modeling period. It is certainly not true that the fundamental risks of Bank of America Corporation have remained constant, given the large number of mergers that have occurred during the 1990 to 2015 period. Assuming away those complications for one sentence, both methods 3 and 4 indicate (using public information only) that Bank of
America would respond better in the Federal Reserve’s severely adverse scenario than it did in the credit crisis of 2006-2010.

References


Appendix A

Merton model background


Merton's model of risky debt views debt as a put option on the firm's value. In this simple model, debt is a discount bond (no coupons) with a fixed maturity. Firm value is represented by a single quantity interpreted as the value of the underlying assets of the firm. The firm defaults at the maturity of the debt if the asset value lies below the promised payment. Using the Black-Scholes technology, an analytic formula for the debt's value can be easily obtained. In this solution, it is well know that the expected return on the value of the firm's assets does not appear. In fact, it is this aspect of the Black-Scholes formula that has made it so useable in practice.

From Merton's risky debt model, one can infer the implied pseudo default probabilities. These probabilities are not those revealed by actual default experience, but those needed to do valuation. They are sometimes called martingale or risk adjusted probabilities because they are the empirical probabilities, after an adjustment for risk, used for valuation purposes. If we are to compare implied default probabilities from Merton's model with historic default experience, we need to remove this adjustment.

Removal of this adjustment is akin to inserting expected returns back into the valuation procedure. To do this, we need a continuous-time equilibrium model of asset returns consistent with the Merton risky debt structure. Merton's [1973] intertemporal capital asset pricing model provides such a structure. We now show how to make this adjustment.

Let the value of the ith firm's assets at time t be denoted by $V_i(t)$ with its expected return per unit time denoted by $a_i$ and its volatility per unit time denoted by $\sigma_i$.

Under the Merton [1974] structure, we have that

$$V_i(t) = V_i(0)e^{\mu t + \sigma Z_i(t)}$$

where

$$\mu_i = a_i - \frac{\sigma_i^2}{2}$$

and $Z_i(t)$ is a normally distributed random variable with mean 0 and variance $t$. 
The evolution of \( V_i(t) \) above is under the empirical probabilities\(^1\). In Merton's [1973] equilibrium asset pricing model, when interest rates and the investment opportunity set\(^2\) is constant, the expected return on the \( i \)th asset is equal to

\[
a_i = r + \frac{\sigma_i \rho_{iM}}{\sigma_M} (a_M - r)
\]

where

\( r \) is the risk free rate\(^3\), the subscript \( M \) refers to the “market” portfolio or equivalently, the portfolio consisting of all assets of all companies in the economy, and \( \rho_{iM} \) denotes the correlation between the return on firm \( i \)'s asset value and the market portfolio.

Using this equilibrium relationship, the drift term on the \( i \)th company’s assets from can be written as

\[
\mu_i = -\frac{1}{2} \sigma_i^2 + (1 - b_i) r + b_i a_M
\]

where

\[
b_i = \frac{\sigma_i \rho_{iM}}{\sigma_M}
\]

is the \( i \)th firm’s beta.

For expositional purposes, we parameterize the expected return on the market \( a_M \) as equal to a constant \( k \) times the risk-free interest rate \( r \), i.e.

\[
a_M = kr.
\]

This is without loss of generality. Using this relation, we have that

\[
\mu_i = -\frac{1}{2} \sigma_i^2 + (1 - b_i + b_i k) r.
\]

In Merton’s risky debt model, the firm defaults at the maturity of the debt if the firm’s asset value is below the face value of the debt. We now compute the probability that this event occurs.

Let \( t \) be the maturity of the discount bond and let \( B \) be its face value. Bankruptcy occurs when \( V_i(t) < B \), formally

\[\text{under the pseudo probabilities, } a_i = r.\]

\[\text{The investment opportunity set is the means and covariances of all the assets’ returns. This implies that the mean and covariances with the market return are also deterministic.}\]

\[\text{In this structure, this is the spot rate of interest on default free debt.}\]
where $N(.)$ represents the cumulative normal distribution function.

We see from this expression that default probabilities are determined given the values of the parameters $(B, V_i(0), \sigma_i, r, \sigma_M, \rho_M, k)$. We discuss the estimation of these parameters in the next section.

**Empirical Estimation of the Default Probabilities**

As mentioned previously, this estimation is based on First Interstate Bancorp bond data. The time period covered is from January 3, 1986 to August 20, 1993. Over this time period, the bank solicited weekly quotes from various investment banks regarding new issue rates on debt issues of various maturities. The two-year issue is employed in the subsequent analysis, see Jarrow and van Deventer [1998] for additional details.

To estimate the default probabilities, we need to estimate the seven parameters $(B, V_i(0), \sigma_i, r, \sigma_M, \rho_M, k)$.

The face value of the firm’s debt $B$ can be estimated using balance sheet data. To do this, we choose $B$ to be equal to the total value of all of the bank’s liabilities, compounded for two years at the average liability cost for First Interstate.

The firm value at time 0, $V_i(0)$, and the firm’s volatility parameter $\sigma_i$ are both unobservable. This is one of the primary difficulties with using Merton’s model and with the structural approach, in general. For this reason, both the market value of First Interstate’s assets and the firm’s asset volatility were implied from the observable values of First Interstate common stock and First Interstate’s credit spread for a two-year straight bond issue. We choose those values that minimized the sum of squared errors of the market price from the theoretical price, see Jarrow and van Deventer [1998] for details.

The spot rate $r$ is observable and the volatility of the market portfolio $\sigma_M$ can be easily estimated from market data. As a proxy for the market portfolio, we used the S&P 500 index. The volatility of the S&P 500 index over the sample period was 15.56%.

Lastly, this leaves the two parameters: the expected return on the market and the correlation between the firm’s assets and the market portfolio. As both of these quantities are unobservable and arguably difficult to estimate, we choose them as “free” parameters. That is, we leave these values unspecified and estimate default probabilities for a range of their values. The range of values for the correlations employed is from -1 to 1, and the range of ratios for the market return to the risk free rate is from $k = 1$ to $k = 5$. 
Finally, in order to compare the derived 2-year default probabilities with the observable credit spread, which is quoted on an annual basis, we convert the actual 2-year default probability to a discrete annual basis using the following conversion formula

\[ \text{probability[annual]} = 1 - \sqrt{1 - \text{probability[twoyear]}}. \]

The following chart provides these annualized default probabilities on November 17, 1989.4

<table>
<thead>
<tr>
<th>Date</th>
<th>Actual Credit Spread</th>
<th>1.220%</th>
<th>5.00</th>
<th>4.00</th>
<th>3.00</th>
<th>2.00</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>11/17/89</td>
<td>Correlation of Asset Returns with Market Returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.06%</td>
<td>0.49%</td>
<td>2.57%</td>
<td>9.09%</td>
<td>22.83%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>0.49%</td>
<td>1.76%</td>
<td>5.06%</td>
<td>11.79%</td>
<td>22.83%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>2.57%</td>
<td>5.06%</td>
<td>9.09%</td>
<td>14.99%</td>
<td>22.83%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>9.09%</td>
<td>11.79%</td>
<td>14.99%</td>
<td>18.67%</td>
<td>22.83%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>22.83%</td>
<td>22.83%</td>
<td>22.83%</td>
<td>22.83%</td>
<td>22.83%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.25</td>
<td>43.08%</td>
<td>37.64%</td>
<td>32.38%</td>
<td>27.42%</td>
<td>22.83%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.50</td>
<td>64.78%</td>
<td>54.16%</td>
<td>43.08%</td>
<td>32.38%</td>
<td>22.83%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.75</td>
<td>82.14%</td>
<td>69.70%</td>
<td>54.16%</td>
<td>37.64%</td>
<td>22.83%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1.00</td>
<td>92.68%</td>
<td>82.14%</td>
<td>64.78%</td>
<td>43.08%</td>
<td>22.83%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Surprisingly, one can see the wide range of default probabilities possible as we vary the free parameters. Across the entire table, the annualized default probability varies between 0.06% to 92.68%. For a correlation of 1 (the typical asset)5, the range in default probabilities for a market return to risk free rate ratio between \( k = 2 \) and \( k = 3 \) is 9.09% to 2.57%. These ranges in default probabilities are quite large and they cast doubts upon our abilities to credibly validate the model using historical default frequencies.

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4 This date was chosen because it was the mid-point of the First Interstate data set.
5 The typical asset has a beta of 1. If the volatility of the market and the firm’s asset are equal, then the correlation is one as well.