

Risk Management Models: Construction, Testing, Usage

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Abstract

Financial risk management models are alleged to have helped cause the 2007 credit crisis. There is some truth to these concerns. Risk management models were used wrongly prior the crisis and they are still being misused today. The purpose of this paper is to critically analyze risk management models - construction, testing, and usage. In the process we show that there are two common misuses of these models associated with calibration and hedging "the greeks." In particular we show that: (i) vega hedging with a calibrated Black-Scholes option pricing model is a nonsensical procedure, (ii) using the implied default probabilities from a structural credit risk model to price credit default swaps is erroneous, and (iii) using the default correlations obtained from credit risk copula models for computing value-at-risk measures leads to misspecified estimates. We provide alternatives to these misuses and a roadmap for the proper usage of risk management models.

1 Introduction

The 2007 credit crisis was a wake-up call with respect to model usage. It has been alleged that the misuse of risk management models helped to generate the crisis. For example, "*The turmoil at AIG is likely to fan skepticism about the complicated, computer-driven modeling systems that many financial giants rely on to minimize risk.*"¹ Even the financial wizard himself, Warren Buffet, is skeptical about models "*All I can say is, beware of geeks...bearing formulas.*"²

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¹Wall Street Journal, Monday Nov 3, 2008, *Behind AIG's Fall, Risk Models Failed to Pass Real-World Test.*

²Wall Street Journal, Monday Nov 3, 2008, *Behind AIG's Fall, Risk Models Failed to Pass Real-World Test.*

Unfortunately there is some truth to these concerns. In numerous cases, models were used wrongly by the financial industry prior to the 2007 credit crisis, and some are still being misused today. Unfortunately, we cannot just blame the financial industry for this misuse. These techniques are often presented without adequate explanation in the standard textbooks on derivatives (see Chance [5], Hull [11], Jarrow and Turnbull [16], Kolb [17], Whaley [21], Rebonato [19]).

Statistician George E.P. Box ([3], p.74) said "Remember that all models are wrong: the practical question is how wrong do they have to be to not be useful." Alternatively stated: all models are approximations, and when is the approximation too wrong to be useful? The purpose of this paper is to answer this question. This paper critically analyzes risk management model construction, testing, and usage.

We consider two types of models: theoretical and statistical. Theoretical models capture *cause and effect* while statistical models identify *correlations*. In constructing models, we characterize a model's assumptions as being one of two types: robust or critical. Robust assumptions are those which if changed slightly, only change the model's results slightly as well. Critical assumptions are those for which this is not the case. In testing models we argue that one needs to test the model's implications and its assumptions, especially the critical assumptions. If either of these are rejected, the model should not be used. With respect to model implementation, we show that there are two common misuses of risk management models related to calibration and hedging of the greeks. In particular, we show that: (i) vega hedging with a calibrated Black-Scholes option pricing model is a nonsensical procedure, (ii) using the implied default probabilities from a structural credit risk model to price credit default swaps is erroneous, and (iii) using the default correlations obtained from credit risk copula models for computing value-at-risk measures leads to misspecified estimates. We provide alternatives to these misuses and a road map for the proper usage of risk management models.

An outline of this paper is as follows. Section 2 presents a characterization of models and explains how models should be tested. Sections 3 and 4 analyze the two common misuses of risk management models. Section 5 synthesizes the previous insights with a discussion on how to properly use models, and section 6 concludes.

2 What is a Model

A model is a mathematical simplification of a real phenomena for the purpose of generating implications and predictions. For this paper we concentrate on risk management models, but much of the following discussion applies more generally. Although obvious, it is important to state the fact that a model is equivalent to its assumptions. And, in turn, the assumptions generate a set of necessary implications. Consider the following exhibit in this regard.

$$\left[\text{Model} \iff \begin{cases} \text{assumption 1} \\ \vdots \\ \text{assumption } N \end{cases} \Rightarrow \begin{cases} \text{implication 1} \\ \vdots \\ \text{implication } M \end{cases} \right]$$

Exhibit 1: Abstract Representation of a Model

As depicted in Exhibit 1 a model is equivalent to its assumptions (N in this exhibit). This is a tautology. A model is constructed to generate a set of implications (M in this exhibit). If one generates enough implications, then they become both necessary and sufficient conditions for the assumptions. In this case, the model is also equivalent to its implications. But, it is rare to find such a model in practice. Hence, the usual situation is as depicted in Exhibit 1.

2.1 Two Types of Models

First, we need to step back and decompose models into two types: (1) statistical and (2) theoretical. A statistical model identifies patterns - *correlations* - in historical data that one hopes will continue into the future. A theoretical model is based on economic reasoning to understand *cause and effect*. Theoretical models have parameters that are estimated from historical data. To help explain the difference between these two types of models, it is easiest to use an example from derivatives pricing.

Example 1 *Call Option Valuation.*

Consider a model to price a European call option on a stock. We illustrate both a theoretical and statistical model.

The Black-Scholes model is a theoretical model for the call's price. Its inputs are the stock price, strike price, maturity date, and volatility. The output is the option value. The model forms a relation between the inputs (cause) and the output (effect). Generalizations of the Black-Scholes model for stochastic volatilities and jumps are all theoretical models.

A simple statistical model for pricing an option can be generated as follows. Using historical option prices, compute the average call price conditioned on the stock price, strike price, time to maturity, and the stock's volatility to obtain the option's value. The historical correlations between the inputs and the output are captured by computing the conditional average. A more sophisticated class of statistical models are those based on time series models (e.g. GARCH) to estimate (predict) an option's implied volatilities. These will be discussed in more detail below.

Both types of models are useful with different costs and benefits. The benefits of theoretical models are that they attempt to understand the causes of the implications, but they are often complex and hard to get right. The benefits of statistical models are that they are often easy to construct, but they do not explain the underlying causes of the patterns. Of course, if the patterns change, then the statistical model will no longer be useful. If the theoretical model captures this cause, then it will still apply.

2.2 Two Types of Assumptions

There are two types of assumptions with respect to any model: robust and non-robust (critical assumptions). A *robust assumption* is one where the implications of the model only change slightly if the assumption is modified only slightly. This corresponds to a "continuity" in the topology of the model's structure. In contrast, a non-robust or critical assumption is one where the implications of the model change discretely if the assumption is only changed slightly. This corresponds to a "discontinuity" in the topology of the model's structure.

This distinction is important because since models are approximations of a complex reality, we may not get the assumptions exactly correct. With robust assumptions, we do not need to worry too much. We need to be careful, however, with the critical assumptions. If we get the critical assumptions wrong, by just a little, the implications completely change. Of course, for large errors in robust assumptions, the implications will also change significantly, despite the "continuity." So, even the robust assumptions are important in model construction.

There is only one way to determine if an assumption is critical or robust. This is to extend the model, relax the assumptions, and determine analytically if the implications change continuously or not. This determination is a key purpose for generalizing models.

Example 2 *Derivative Pricing Model Assumptions.*

The standard derivative pricing models assume frictionless markets (no transaction costs, no trading restrictions or short sale constraints), competitive markets, a stock price process, and no profitable arbitrage opportunities. With respect to these assumptions, we can partition them into robust and critical assumptions as follows:

- *Frictionless markets - no transaction costs and temporary quantity impacts on the price process are robust assumptions (see Broadie, Cvitanic and Soner [4], Jarrow and Protter [15]). A small transaction cost or temporary quantity impact on the price will not change pricing nor hedging dramatically. These frictions create a wedge around the frictionless price.*
- *Frictionless markets - no trading restrictions or short sale constraints is a critical assumption. Its violation makes the market incomplete and eliminates certain hedging strategies. Derivative pricing changes completely. An extreme form of this assumption is that the underlying asset doesn't trade, which implies hedging is impossible and the market is incomplete.*
- *Competitive markets (or no permanent quantity impacts on the price process) is a critical assumption. Violations of this assumption generate market manipulation. Market manipulation causes a complete reversal of pricing and hedging (see Jarrow [12]).*
- *The assumed stock price process is a robust assumption as long as a small perturbation is defined as one which retains market completeness. As is*

well known, changing from a complete to an incomplete market dramatically changes pricing and hedging. Such a change in a stock price process can be viewed as non-local change in a robust assumption.

- *No profitable arbitrage opportunities is a critical assumption. If a market contains an arbitrage opportunity, then market prices need not satisfy any of the implications of the derivative valuation model.*

2.3 Testing Models

To test a model, the usual method is to test the implications.³ However, by testing the implications, one can really only reject a model. This occurs if any implication is violated. A model cannot be proven if all the implications are accepted, unless they are a sufficient set of implications - which is almost never the case. Unfortunately, this implies that the usual method for testing a model is problematic.

But, all is not lost. There is an alternative approach that one can use to test a model - test the assumptions. In this regard there are two opposing views. Milton Friedman [10] said "a theory cannot be tested by comparing its "assumptions" directly with "reality." Indeed, there is no meaningful way in which this can be done....the question of whether a theory is realistic 'enough' can be settled only by seeing whether it yields predictions that are good enough for the purpose in hand or that are better than predictions from alternative theories." More recently, Robert Aumann ([1], p.388) also echoes these views. In contrast, Robert Solow [20] said "All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A 'crucial' assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect."

We agree with Solow's argument. Since the assumptions are necessary and sufficient conditions for the validity of a model, one can reject a model if any assumption is violated, and one can validate a model if all the assumptions are accepted. However, there is still a practical problem with this alternative approach. Not all of the model's assumptions may be observable and, therefore, testable. An example is in equilibrium asset pricing theory where a traders' utility functions and endowments are not observable.

Although there is no complete solution to this non-observability problem, it is sometimes possible to overcome this non-observability problem in model construction. When constructing a model, it might be possible to impose assumptions only on observables. If not, a practical solution is to construct a model such that all critical assumptions are observable, and only (a subset of) the robust assumptions are not.

³To test a model's implications or assumptions, the standard hypothesis testing procedures that incorporate errors in observations need to be invoked.

This leads to the following rules for testing and using a model.

- $$\left[\begin{array}{l} (i) \text{ Test all implications,} \\ (ii) \text{ test all critical assumptions,} \\ (iii) \text{ test all observable robust assumptions, and} \\ (iv) \text{ "believe" all nonobservable robust assumptions.} \\ \text{Reject unless conditons (i) - (iv) are accepted.} \end{array} \right]$$

Exhibit 2: Method for Validating a Model

Two examples help to clarify this validation procedure.

Example 3 A Lucky Charm.

Consider a baseball player that can only hit fast balls and nothing else. By chance, let us also suppose that the first two pitchers pitching against the hitter are fast ball throwers.

Suppose that for pitcher one this baseball player wears a pair of socks. Of course, he hits a home run. For pitcher two, he wears the same pair of socks again, and he hits another home run. Observing this correlation, the socks become his "lucky charm."

The model is "wearing the socks generates a home run." The player observes the implication "a home run" is true and accepts the model. This is accepting a model based on only testing the implications.

By construction, it is clear that the first time he wears the socks and a curve ball pitcher pitches, he will strike out. In which case the model will be rejected.

Although simple, this example can also be used to show the consequences of using such a lucky charm model in industry. The continued coincidence of a successful outcome and the application of the model gives the user increased over-confidence in the model's validity. If the other implications of the model are then used as well, problems can arise. At some point the luck will run out. This analogy applies to the credit rating agencies and AIG's use of the copula model for gauging a Credit Default Obligations (CDO) default risk (see Bluhm and Overbeck [2] and the discussion below on calibration). A related and continuing example of the misuse of credit risk models is with respect to the debate between using structural versus reduced form models (see Jarrow [14] for a recent discussion of this debate).

Example 4 Credit Risk Models.

In credit risk modeling there are two model types: reduced form and structural. A crucial assumption in each is the ability to trade the underlying assets (see Jarrow [13] for a review). A violation of this assumption causes the implications of the model to completely change. This is a violation of the no trading restrictions assumption of derivative pricing models discussed earlier.

*For the reduced form models the traded asset is a particular issue of the firm's risky debt. For structural models the traded assets are **all** of the firm's assets (equivalently, **all** of the firm's liabilities and equity).*

One can test this critical assumption for both models by direct observation. For reduced form models the assumption is accepted, for structural models it is rejected. Hence, structural models should be rejected based on this evidence alone.⁴

3 Theoretical Model Parameter Estimation

Now that we understand how to test a model, we turn to the estimation of a theoretical model's parameters. There are two approaches: historical estimation and implicit estimation or calibration. Historical estimation is using historical data and standard statistical procedures to estimate the model's parameters. Historical estimation is always a correct procedure. Calibration is estimating the model's parameters by equating the model's price to the market price. Calibration may or may not be a correct estimation procedure, depending upon the circumstances at hand. There are two cases to consider in this regard based on a previous testing of the model: the model is accepted or the model is rejected.

If the theoretical model is accepted, then calibration is equivalent to historical estimation. In this case, calibration provides an alternative and perhaps easier estimation procedure. Of course, when using calibration in this manner, one needs to avoid a potential circularity in the pricing of derivatives.

If the theoretical model is rejected, then calibration is used for an alternative purpose. In this case calibration is used to overcome pricing errors in the model (based on historical estimation procedures). When used in this manner, a calibrated model can be easily misused. To understand this misuse, we need to step back and discuss this use of calibration in more detail.

3.1 The Transformation

When a theoretical model is rejected, calibration transforms a theoretical model into a statistical model. To prove this claim, we need to introduce some notation. Consider a theoretical model $f : R^{K+1} \rightarrow R$ representing a derivative's price, i.e.

$$p_u = f(q, u)$$

where $u \in R^K$ is a K - vector of known constants, q is the parameter to be estimated, and p_u is the market price. To help with the intuition, one can think of p_u as a European call option's price, u represents the (stock price, strike

⁴Of course, various scholars have tested implications of both models. In this regard, the evidence rejects the implications of simple forms of the structural model and accepts the reduced form models (see Jarrow [14] for a review). This has lead various authors to seek generalizations of the simple structural model whose implications are more consistent with the evidence. However, these generalizations still maintain the same critical assumption, which can be directly tested and rejected.

price, time to maturity), q is the underlying stock's volatility, and $f(q, u)$ is the Black-Scholes model. Although, the formulation is purposely much more general and abstract.

By assumption, we are considering the case where the theoretical model has been rejected. That is, given a historical estimate \hat{q} for the parameter, it is known that

$$p_u^{obs} \neq f(\hat{q}, u)$$

where p_u^{obs} corresponds to the observed market price.

Despite this rejection, one still wants to use the model. Is this still possible? The answer is yes. Calibration can be used for this purpose. Here is how. Given is a set of market data $\{p_1^{obs}, u_1\}$. Using this data, one estimates q^{cal} such that $p_1^{obs} = f(q^{cal}, u_1)$, i.e.

$$q^{cal} = f^{-1}(p_1^{obs}, u_1)$$

where q^{cal} is the calibrated parameter estimate. Then, given this estimate q^{cal} , a model for the derivative's price p_u for $u \neq u_1$ is given by

$$p_u = f(f^{-1}(p_1^{obs}, u_1), u).$$

As given by this expression, it is now easy to see that this is a *statistical model* for the derivative's price based on non-linear estimation. Although the choice of the functional form for the non-linear estimation is based on a theoretical model, the theoretical model itself is rejected. Hence, it is no longer a theoretical model. This simple observation is not well understood.

3.2 Misuses of Calibration

We have just proven that a rejected and calibrated theoretical model is, in fact, a statistical model formulated to match market prices p . This is a legitimate use of calibration. However, such a statistical pricing model should be used for no other purpose. This follows precisely because in this case, the need for calibration is predicated on the fact that the theoretical model has been rejected. Using the calibrated model to obtain the additional implications of the theory is no longer valid and constitutes a misuse of the procedure. We illustrate three such common misuses and comment on correct alternatives to be used instead.

Example 5 Implied Black-Scholes Volatilities.

The first example of calibration and its misuse is with respect to the Black-Scholes model and implied volatilities.

First, it is well-known (see Jarrow and Turnbull [16]) that for equity options, when using historical volatility estimates, one can reject the Black-Scholes model.

Hence, to match market prices, calibration is used. Given the unknown parameter is the equity's volatility, the use of implied volatilities has arisen. When estimating implied volatilities, a smile or sneer in the strike price and the time to maturity profile is regularly observed.⁵

⁵Of course, this observation also provides a further rejection of the Black-Scholes model.

Implied volatilities are useful in constructing a statistical model for estimating option prices yielding a calibrated Black-Scholes model. With respect to calibrated Black-Scholes values, one can also use sophisticated time series methods to predict future implied volatilities (e.g. GARCH). This is a valid use of calibration.

The misuse occurs, however, if one still attempts to use the Black-Scholes model with implicit volatilities to do hedging (see Chance [5], Hull [11], Kolb [17], Whaley [21]). This use is invalid because the theoretical Black-Scholes model has been rejected. Not surprisingly, in this use, it has been observed that delta (plus gamma) hedging doesn't work. Consequently, the practice of vega hedging has arisen (vega hedging is another misuse of risk management models that is separately discussed below). These same incorrect procedures are also common market practice when pricing caps and floors using a calibrated LIBOR market model, see Rebonato [19].

There are two correct alternatives that one can use for hedging in this circumstance. First, one can develop a more general theoretical option model incorporating stochastic volatilities. Alternatively, one can use a statistical hedging model. Both of these alternatives are described in greater detail below in the section on hedging the greeks.

Example 6 Implied Default Probabilities using Merton's ([18]) Model.

A second example of the misuse of calibration is with respect to structural credit risk models and implied probabilities of default.

As discussed previously, it is well-known that structural credit risk models assume that all of the firm's assets trade and their values are observable (see Jarrow [14]). It is also an accepted fact that all of the firm's assets do not trade and that the firm's value (and volatility) is unobservable. These two facts reject the structural model's critical assumptions, and hence the model.⁶

Nonetheless, the model is still desired to be used. The models unobservable inputs are the firm's asset value and volatility. To estimate these two inputs calibration is used. The calibration uses observable stock prices and the stock price's volatility to obtain estimates of the firm's value and volatility. This is often done by minimizing the error between the market observed stock price and volatility versus the model's values. As a result of this estimation, the calibrated structural model is useful for obtaining a non-linear estimate of the stock's price and volatility. This is a valid use of calibration.

However, in industry practice, the model is still incorrectly used as a theoretical model for a second purpose. The second purpose is to estimate the firm's probability of default (or the distance to default) for use in pricing credit default swaps (CDS). The probability of default estimates obtained from using a calibrated structural model are misspecified and should not be used.⁷

⁶In addition, as mentioned previously, the majority of the empirical evidence rejects the simpler structural models.

⁷There is also an abundance of academic evidence rejecting these default probabilities as good estimates of the true underlying default probabilities (see Jarrow [14]).

There are at least two correct alternatives for estimating default probabilities. One is to use hazard rate estimation procedures based on historical default data (see Chava and Jarrow [6]). A second is to use a reduced form model that is not rejected by the data.

Example 7 Implied Default Correlations using a CDO Copula Model.

A third example of the misuse of calibration relates to the copula model for pricing collateralized debt obligations (CDOs) and implied default correlations. The standard CDO copula pricing model (see Bluhm and Overbeck [2]) depends on the correlations between the underlying firm's asset returns (pairwise). For simplicity, it is often assumed that these correlations are the same across all firms, called ρ .⁸

As with the structural model, the CDO copula model assumes that all the assets of the firm trade and are observable. As discussed in the previous example, these assumptions are not true and hence the model is formally rejected.⁹

Nonetheless, the model is still desired to be used to price CDOs. For this purpose, an implied correlation coefficient can be obtained by calibrating the CDO's model price to the market price. When calibrated in this manner, the CDO copula model generates a valid statistical model for pricing CDO's.

The misuse occurs when the model is still viewed as a theoretical model and employed for a second purpose. The second purpose is to estimate the probability distribution for the losses of the CDO's underlying collateral pool for use in value-at-risk computations. Since the theoretical CDO copula model is rejected, its use to estimate this loss distribution is invalid. Unfortunately, it was this misuse that partially contributed to the incorrect belief prior to the 2007 credit crisis that subprime CDOs were safe. This misuse was the basis of the credit rating agencies incorrect ratings of subprime CDOs (see Crouhy, Jarrow, Turnbull [7]).

As in the previous example, there are at least two correct alternatives for estimating default probabilities. One is to use hazard rate estimation procedures based on historical default data. These procedures also provide an estimate for default correlations across the different credit entities. A second is to use a reduced form model that is not rejected by the data.

4 Hedging the Greeks

Another misuse of derivative pricing models is related to "hedging the greeks." Some derivative model's greeks can be hedged correctly and some cannot. The greeks that can be hedged correctly are those related to stochastic processes as they represent risks within the derivative model. The greeks that are hedged incorrectly are those that are constant or deterministic parameters and that do not represent risks within the model. The need to hedge a constant parameter

⁸Some minor generalizations of this simplifying assumption have been considered (see Jarrow [13]). Nonetheless, the logic as discussed here still applies.

⁹Of course, since the assets are unobservable, we cannot estimate this correlation directly.

implies that an assumption underlying the theoretical model is rejected, the constant parameter is stochastic, and therefore the model is rejected and should not be used.

It is easiest to explain this misuse using an example that is in most derivative textbooks (see Chance [5], Hull [11], Kolb [17], Whaley [21], Rebonato [19]) and that is quite common in industry usage: vega hedging or hedging the changing option's volatility when using a calibrated Black-Scholes formula.

Example 8 Vega hedging a calibrated Black-Scholes model.

Let $p = f(S, \sigma)$ represent the Black-Scholes model where σ is the stock's volatility, assumed constant, p is the option's price, and S is the stock price. Here only S is random, hence, there is only one risk in the model. This price risk is embedded within the model via the standard complete market hedging argument and the use of risk-adjusted (or risk neutral) probabilities (see Jarrow and Turnbull [16]).

Hence, as is well known, if the model is correct, then to hedge the option's risk, we need to only hedge changes in S using the stock's delta. However, if we cannot trade quickly enough, gamma hedging is appropriate to hedge changes in the delta as well. As these hedges relate to price risk, which is incorporated into the model's construction, this usage is correct.

But, in practice and in contradiction to the model's assumptions, we observe that the stock's volatility σ is random. And, we observe that delta and gamma hedging are insufficient to remove the risk from a hedged stock and option position. Of course, this formally rejects the model.

Nonetheless, the Black-Scholes model is still used in an attempt to hedge the volatility risk. To "fix" this hedging error, a Taylor series expansion is applied to the Black-Scholes formula writing the change in the option's price as a linear function of the option's delta times ΔS plus the option's vega $\frac{\partial f}{\partial \sigma}$ times $\Delta \sigma$. The goal is to use this linear expansion of the Black-Scholes formula to hedge changes in the volatility ($\Delta \sigma$).

But, this linear approximation is nonsensical because the volatility is a constant ($\Delta \sigma \equiv 0$) in the model's construction. There is no volatility risk priced within the Black-Scholes model, so the vega ($\frac{\partial f}{\partial \sigma}$) can give no indication of how changes in the volatility affect the call's price.

We can understand the magnitude of the error introduced in vega hedging a calibrated Black-Scholes model by considering a more general call pricing model that includes volatility risk and where vega hedging is correct.

Example 9 Vega Hedging in a Stochastic Volatility Model.

This example shows the magnitude of the errors that occur in vega hedging when using the Black-Scholes formula.

Let us consider a more general stochastic volatility option pricing model where σ_t is the stock's stochastic volatility at time t . In this model, the stock price and volatility are both correlated random processes. It can be shown (see Eisenberg and Jarrow [8], Fouque, Papanicolaou and Sircar [9]) that the true

call option price is

$$C = \int_0^\infty f(\Theta_0)h(\Theta_0 | \sigma_0)d\Theta_0$$

where $f(\cdot)$ is the BS formula, T is the option's maturity date, $\Theta_0 = \int_0^T \sigma_s ds$, and $h(\Theta | \sigma_0)$ is the risk neutral probability of Θ_0 which is conditional on σ_0 .

We see that the call option's true price is a weighted average of Black-Scholes values. The correct vega hedge ratio is therefore

$$\frac{\partial C}{\partial \sigma_0} = \int_0^\infty \left[\frac{\partial f(\Theta_0)}{\partial \Theta_0} \frac{\partial \Theta_0}{\partial \sigma_0} h(\Theta_0 | \sigma_0) + f(\Theta_0) \frac{\partial h(\Theta_0 | \sigma_0)}{\partial \sigma_0} \right] d\Theta_0.$$

It is clear that this is not equal to the Black-Scholes vega, i.e. $\frac{\partial f(\sigma_0)}{\partial \sigma_0}$.

A simple example illustrates this absurdity even more clearly than the above use of vega hedging a calibrated Black-Scholes model.

Example 10 Delta Hedging in a Deterministic Model.

This example shows how hedging a change in an option pricing model's constant parameter leads to nonsensical results.

Let us use a model where we assume that the stock price S is non-random and (because it is riskless) that it grows at the risk free rate r . Then, the call option's value is

$$\begin{aligned} c &= \tilde{f}(S_t) \\ &= \max\{S_T - K, 0\}e^{-r(T-t)} \\ &= \max\{S_t - Ke^{-r(T-t)}, 0\} \end{aligned}$$

where $\tilde{f}(\cdot)$ is the call option's value with strike price K and maturity T . We note that there are no risks in this call option model.

Of course, we observe that the stock price S is random, and that the model's price does not match market prices. Formally, the model is rejected.

Nonetheless, we want to still use the model. Hence, we calibrate the model to the market price by selecting that stock price which equates the model's price to the market price. This yields an "implied stock price" (S_{imp}). Of course, S_{imp} will not equal the market price (analogous to the implied volatility not equaling the historical volatility in the Black-Scholes model). The calibrated option model now is transformed to a statistical model, only useful for pricing call options.

But, let's still use this calibrated option model to hedge the stock price movements by "hedging the greek delta." We do this in the standard way by using a Taylor series expansion on the call's model price in ΔS times the delta $\frac{\partial \tilde{f}}{\partial S}$.

It is easy to see that the delta in this Taylor series expansion is either 1 or 0 depending on whether the stock price is in- or out-of-the-money, respectively. In words, one needs to either sell the entire stock or don't hedge. It is self-evident that this hedge will not work. This is true despite the fact that the calibrated option model is purposely constructed to provide reasonable option prices. Hedging the "greek" does not work here precisely because the original theoretical option model had no stock price risk embedded within it.

5 How to Properly Use a Model

This section collects and summarizes the previous insights regarding how to properly use a risk management model. If followed, these simple rules will enable the model's user to make better decisions with the model than without.

Model Construction There are "good" and "bad" theoretical models. A good model's critical assumptions can be tested using observable market data, although its robust assumptions need not be. If the robust assumptions are not testable using observable market data, they should be intuitively valid. A bad model is one that does not satisfy these conditions. One should only use a good model.

Testing Models To test a theoretical model, one should follow the rule given in Exhibit 2. One should test all the model's implications, all the critical assumptions, all the observable robust assumptions, and be convinced that the nonobservable robust assumptions are approximately true. If any of these tests are rejected, the model is false and should not be used.

In addition, as time passes and new data is collected, the model should be continually retested for its validity. This is especially true of the critical assumptions, which if invalidated by an unexpected shift in the structure of the economy, cause the model's implications to no longer apply.

Parameter Estimation and Calibration Estimating the parameters of a theoretical model using a historical estimation procedure is always correct. Implicit estimation, or calibration, has two uses. If the theoretical model is accepted, calibration is equivalent to historical estimation, and calibration is a correct alternative to historical estimation. If the theoretical model is rejected, then calibration transforms a theoretical model into a statistical model. In this case calibration should only be used to estimate the derivatives price. Since the theoretical model is rejected, its other implications, including determining hedge ratios, are invalid.

This does not imply, however, that one cannot use a statistical model to determine hedge ratios. For example, in the case of a calibrated Black-Scholes model, one can run a simple regression

$$\begin{aligned}\Delta p_t &= \beta_0 + \beta_1 (\text{delta} \times \Delta S_t) \\ &\quad + \beta_2 (\text{gamma} \times [\Delta S_t]^2) \\ &\quad + \beta_3 (\text{vega} \times \Delta \sigma_t) + \varepsilon_t\end{aligned}$$

where $\beta_0, \beta_1, \beta_2, \beta_3$ are constants, p is the derivative's market price, S is the stock's price, σ is the volatility, and ε is the estimation error. Delta, gamma, and vega come from the Black-Scholes model. This purely statistical hedging model uses the partial derivatives of the Black-Scholes model to adjust for the inputs to the model (stock price, strike price, maturity, volatility). As estimated

herein, the hedges are based on patterns in the data and not a theoretical model. If the patterns change, the statistical hedging model will not longer work. This, of course, is the problem with statistical models.

Hedging the Greeks When using a theoretical model, only the risks originally embedded within the model's construction can be hedged using the greeks. The deterministic parameters of the model are not priced within the model, and using a Taylor series expansion to hedge their randomness is invalid. The need to hedge these risks invalidates the original theoretical model. A more general theoretical model need to be constructed, or a statistical hedging model needs to be employed.

6 Conclusion

The best way to understand models and their use is to consider an analogy. Models are analogous to medical prescription drugs. Prescription drugs have great medical benefits if used properly, with educated use. If used wrongly, however, prescription drugs can have negative consequences, even death. Because prescription drugs can cause death if used wrong, this does not mean that we should stop using them. It does mean, however that we need educated use. In fact, in reality, prescription drugs should probably be used more because they save and prolong lives. *The same is true of models.*

Financial markets have become too complex to navigate without risk management models. Determining a price - fair value - is not an issue because in many cases expert judgment can provide reasonable estimates. But,

- *there is no way to hedge a portfolio, i.e. determine hedge ratios, without a model,*
- *there is no way to determine the probability of a loss (risk measures) without a model, and*
- *there is no way to price a derivative in a illiquid market without a model.*

These issues are at the heart of risk management. Hence, financial risk management models are here to stay.

Models if used properly, help decision making. Models if used improperly, generate bad decisions and lead to losses. This doesn't mean that we should not use models. Quite the contrary. It only means that we need more educated use of models. Had models been properly used before the crisis, the 2007 credit crisis, perhaps the crisis would have not occurred.

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