The Pricing of Risky Debt when Interest Rates are Stochastic

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Abstract

We derive a closed form solution for the valuation of risky discount debt when interest rates are stochastic. We extend Merton's [1974] approach to risky discount debt valuation under constant interest rates to the stochastic interest rate environment of Vasicek [1977]. Business risk is modelled as a geometric Brownian motion that is correlated with interest rates. We derive and explore the relationship between the credit premium and the term premium of corporate debt. In a banking environment, the results can also be applied to analyze (a) the allocation of capital to banking activities of varying credit risk and interest rate risk, (b) the measurement of relative capital adequacy when compared to peer banks, and (c) the interest rate risk-minimizing funding strategy.

1. Introduction

The determination of a corporation's cost of debt capital is an important exercise for both corporate treasurers and bankers. Term structure models have been widely applied to determine the term premium component in bond prices (Vasicek [1977], Cox, Ingersoll and Ross [1985], and Longstaff and Schwartz [1992] represent a short list). Relatively less academic attention has been paid to the credit component of the cost of capital, however. Option-based models have been applied to corporate debt in order to understand the effects of credit variables on the spread. For example, a pioneering study by Merton [1974] adapts the Black-Scholes [1973] option pricing model to the pricing of risky discount debt. In spite of the assumption of a constant interest rate, Merton's model yields important insights into the determinants of the credit spread. This paper generalizes Merton's risky debt pricing model to allow for stochastic interest rates. In this model, we examine the combined effect of term structure variables and credit variables on debt pricing.

To emphasize the marginal impact of the term structure volatility and correlation effects, we retain the structure of Merton's model, but generalize it to allow for stochastic interest rates as in Vasicek [1977]. In Merton's model, the value of corporate assets follows a stationary lognormal process, and interest rates are assumed constant. In the Vasicek model, interest rates follow a mean-reverting process with constant volatility. Surprisingly, Merton's [1973] earlier valuation of options with stochastic interest rates and time-varying volatility can then be applied to find a closed-form expression for the value of risky debt.

Our bond pricing equation yields comparative static results that are consistent with Merton's: the credit premium is increasing in the face value of the debt and the volatility of the assets, and decreasing in the value of corporate assets. Supplementing Merton's observations, we find that the credit spread is an increasing function of the (risk-free) term structure volatility for reasonable parameter values; however, term structure effects can cause the sign of the derivative to change. We also find that changes in the correlation between interest rates and asset value may have a positive or negative impact on the credit spread; the comparative statics are parameter-sensitive. For reasonable parameter values, as the correlation increases, the credit spread increases.

In addition to the pricing issues mentioned above, this analysis allows us to explore a series of critical issues faced by managers of financial institutions and by financial managers of corporations. These questions are fundamental to their task: How does the correlation between a bank's credit risk and interest rate movements affect its borrowing cost? What maturity debt (or face value) should a corporate treasurer issue to minimize fluctuations in the value of the corporation's stock price? How much capital should be allocated to activities within a banking company which vary both in the absolute degree of credit risk and in the correlation of that risk with movements in interest rate risks?

The next section of this paper outlines the framework developed by Merton and summarized by Ingersoll [1987]. Section 3 extends Merton's work to include the case of stochastic risk-free interest rates. Section 4 discusses the implications of the risky debt pricing formula developed in Section 3 by addressing the issues raised above. Section 5 shows how the formula can be applied to capital allocation and capital adequacy comparisons for commercial banks. Section 6 summarizes our results.

2. Options Theory and the Valuation of Risky Debt: Merton's Model

We start by looking at a simplified case of a bank (or a corporation) whose assets have a market value V. The value of assets V is assumed to be uncertain due to factors such as basic business risk, credit risk, foreign exchange risk, or the price risk of marketable securities held by the organization. We ignore the existence of deposit insurance in some banking markets, such as the United States. We assume that the returns on the bank's assets are instantaneously normal, i.e. that the return on the bank's assets follow the stochastic process

$$\frac{dV}{V} = \alpha dt + \sigma_v dz_1$$

where α and σ are the constant drift and volatility of asset values. This assumption, while commonly applied to corporate assets, is at best an approximation for the stochastic process followed by bank assets. For example, banks own fixed income assets that by construction will not follow geometric Brownian motion processes. For purposes of exposition, we initially assume that the risk-free interest rate is constant; this replicates Merton's results so that they may be compared to our results. In the next section, we extend this approach to the case of stochastic risk-free interest rates. We assume that the bank's assets are financed at time t by the issuance of zero coupon bonds with principal B that is due to mature at time T. We also assume there are no cash distributions to equity until time T. Given the risk inherent in the bank's balance sheet, what should be the pricing on this risky debt? In the words of a bank treasurer, what should be the spread to treasuries (the risk free rate) on the bank's debt?

We begin by assuming perfect markets, free of transaction costs, taxes and informational differences among participants. In this Modigliani-Miller [1958] environment, the market value of the firm is the sum of debt and equity values; the value of the firm is independent of capital structure. We assume that all of the firm's assets can be or will be costlessly converted to cash at time T. If the value of the firm's assets at time T is greater than the principal value of the zero coupon debt B, then the bonds will be paid off in full; otherwise, debtors receive the firm's assets. The value of equity at time T is therefore

$$E = \text{Max}[V_T - B, 0]$$

The equity of the firm is equivalent to a call option on the assets of the firm.

Assuming V can be traded or perfectly replicated, the well-known Black-Scholes call option pricing formula on an asset with value V, volatility σ, time to maturity τ, strike price B, and riskless rate of interest r is

$$Equity\ Value = V\ N(d_1) - B e^{-r\tau} N(d_2)$$

$$where\ d_1 = \frac{\ln(V/B) + (r + \text{‰}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

where N(z) is the cumulative normal density evaluated at z. Since the value of the firm consists only of debt and equity, the value of the risky debt is:

$$D = V\,N(h_1) + B\,e^{-r\tau}\,N(h_2)$$

$$\text{where } h_1 = \frac{\ln(B\,e^{-r\tau}/V) - \text{‰}\,\sigma^2\tau}{\sigma\sqrt{\tau}}$$

$$h_2 = -h_1 - \sigma\sqrt{\tau}$$

The required rate of return on debt is given by

$$r_D = -\frac{1}{\tau}\ln\left(\frac{D}{B}\right)$$

It is well known that the credit spread, r_D-r, is increasing in the face value of the debt and the volatility of the assets, and decreasing in the value of the assets. For a more detailed derivation and discussion of this topic, see Ingersoll [1987] or Uyemura and van Deventer [1992]. The next section extends the analysis to the case where the riskless interest rate is stochastic and is correlated with fluctuations in the value of this underlying asset.

3. The Pricing of Risky Debt with Stochastic Interest Rates

By making use of Merton's [1973] model for the pricing of options with stochastic interest rates, the pricing of risky debt can be extended to the case of stochastic risk free interest rates. Merton's valuation formula for options when interest rates are stochastic is predicated on the assumption that the instantaneous variance of the return on a riskfree zero coupon bond depends only on time to maturity and "is otherwise assumed to be nonstochastic and independent of the level of P," the price of a risk-free bond. This requirement is consistent with the term structure model of Vasicek [1977], but not the models of Cox, Ingersoll and Ross [1985] or Longstaff and Schwartz [1992], for example.

In order to take advantage of Merton's work, we assume that the risk-free term structure is consistent with the Vasicek model. The Vasicek model assumes that the short term riskless interest rate is mean reverting to long run mean γ at speed k, and that its instantaneous volatility (σ_r) is constant:

$$dr = k(\gamma - r)dt + \sigma_r dz_2$$

The Vasicek model suffers from the implicit assumption that at any given time, the future instantaneous interest rates are normally distributed. While this implies the possibility of negative interest rates in future time periods, this liability is offset by the Hull and White [1992] observation that the modified Vasicek model can be used to fit any observable term structure. They note that the Cox, Ingersoll and Ross model [1985], which avoids the possibility of negative interest rates, is unfortunately not sufficiently general to fit all possible term structures. When movements in the short term interest rate take the Vasicek form, the price of a zero coupon bond is priced according to the Vasicek formula:

$$P(\tau) = \exp\left[\frac{1 - e^{-k\tau}}{k}(R(\infty) - r) - \tau R(\infty) - \frac{\sigma_r^2}{4k^3}(1 - e^{-k\tau})^2\right]$$

where

$$R(\infty) = \gamma + \frac{\sigma_r}{k}\lambda - \frac{1}{2}\frac{\sigma_r^2}{k^2}$$

Note that lambda is the market price of risk for such risk-free bonds and must be independent of the bond maturity for no-arbitrage assumptions to prevail. We also assume that the stochastic factors driving the instantaneous returns on bank assets and movements in the instantaneous risk free interest rates are correlated:

$$dz_1 dz_2 = \rho dt$$

Using the same logic as Section 1, the value of risky debt when interest rates are stochastic can be written

$$F = V - VN(h_1) + BP(\tau)N(h_2)$$

where

$$\delta(s) = -\frac{1 - e^{-ks}}{k}\sigma_r$$

$$v^2(s) = \sigma_v^2 + \delta(s)^2 - 2\rho\,\sigma_v\,\delta(s)$$

$$T = \int_0^\tau v(s)^2\,ds$$

$$= \tau\left(\sigma_v^2 + \frac{\sigma_r^2}{k^2} + \frac{2\rho\,\sigma_v\sigma_r}{k}\right) + \left(e^{-k\tau} - 1\right)\left(\frac{2\,\sigma_r^2}{k^3} + \frac{2\rho\,\sigma_r\sigma_v}{k^2}\right) - \frac{\sigma_r^2}{2\,k^3}\left(e^{-2k\tau} - 1\right)$$

and

$$h_1 = \frac{\ln\left(\frac{V}{P(\tau)B}\right) + \text{‰}T}{\sqrt{T}}$$

$$h_2 = h_1 - \sqrt{T}$$

N(z) is the cumulative normal distribution function. $\delta(s)^2$ is the instantaneous variance of the Vasicek model risk-free zero coupon bond with maturity s, $v(s)^2$ is the instantaneous variance of the risky debt function F, and T is the integrated instantaneous variance of the risky debt function F over the life of the risky bond.

This intuitive derivation of the value of risky debt when rates are stochastic can be evaluated more formally by deriving the appropriate partial differential equation for risky debt pricing. The value of risky debt is a function of two stochastic factors V and r. One can use Ito's lemma and the standard no-arbitrage arguments to show that F must satisfy the following partial differential equation:

$$F_t + \frac{1}{2} F_{vv} V^2 \sigma_v^2 + \frac{1}{2} F_{rr} \sigma_r^2 + F_{rv} \rho \sigma_r \sigma_v V + F_r \left[k(\gamma - r) - \lambda \right] - rF + r F_v V = 0$$

At maturity, the value of risky debt F must equal the smaller of the face value of the debt or the value of bank assets:

$$F(\tau = 0) = \min(B,V)$$

One can show that the solution above satisfies the standard partial differential equation and the associated boundary condition.

4. Implications of the Risky Debt Formula

From the point of view of a corporate treasurer or senior manager at a commercial bank, a commonly asked set of corporate finance strategy questions arise. How does leverage impact the "spread", the difference between the yield on the firm's risky debt and the corresponding maturity risk-free bond? How does the volatility of interest rates impact this credit spread? How does the riskiness of the underlying assets impact credit spread? How does the correlation of credit risk with interest rate risk impact financing costs? What maturity debt should be selected to minimize the volatility of the firm's equity? How much capital should be allocated to finance assets of different riskiness so that the cost of debt financing for each asset class will be equal?

We will address each of these issues in turn. We note that the debt being issued by this hypothetical bank or corporation is zero coupon debt with no covenants that would allow the debt holder to trigger bankruptcy no matter what value assets may have prior to the maturity of the debt. We assume that the value of assets V is initially 100 and that debt policy is set by analyzing a debt financing amount F at different maturities, which means that the principal amount B (which effectively includes the future value of interest as well as the up-front amount of "principal" F) will be different for each maturity. The yield on risky debt and the yield on risk-free debt are calculated on a continuous basis, as is the "credit spread", which is equal to the difference between the two rates. The credit spread is algebraically defined as the difference between the continuously compounded (promised) debt yield, and the comparable yield on a zero-coupon bond of the same maturity:

$$\begin{aligned} \text{Credit Spread} &= r_D - r_P \\ &= -\ln(D/B)/\tau + \ln(P)/\tau \\ &= \ln(PB/D)/\tau \end{aligned}$$

The Impact of Leverage

Figure 1 shows the increase in credit spread that results when leverage (the borrowing amount F) is increased from 50 percent of assets to 95 percent of assets assuming σ_r is 0.06, σ_v is 0.11, and the correlation between interest rate movements and the asset returns is 0.3. Given these assumptions, the credit spread is monotonically upward sloping with maturity and with the amount of leverage.

The property that the credit spread increases with the face value of debt is correct regardless of parameter choice. However, for some parameter choices, credit spread declines with debt maturity.

The Impact of Interest Rate Volatility

Given the same base case assumptions and a financing amount F equal to 95, increases in the volatility of the instantaneous risk free interest rate dramatically increase the credit spread shown in Figure 2. This directional result is not universally correct. Changes in interest rate volatility affect the volatility of bond returns through changes in the slope of the term structure and through the correlation of interest rate changes with asset value changes. Of course, for prespecified parameter values, the partial derivative can be signed.

The Impact of Asset Volatility

Figures 3 and 4 show the impact of increasing asset volatility on the credit spread. Figure 3 shows the kind of monotonic increase in credit spread one might expect, both with respect to maturity and the amount of asset volatility. Figure 4 shows clearly, however, that the credit spread may well decrease with maturity if asset volatility is high enough and leverage (represented by F) is also high.

Correlation between Asset Returns and Interest Rates

Figure 5 confirms that the impact of correlation between asset returns and the instantaneous interest rate increases credit spread for the base case assumptions as correlation increases, but it also demonstrates that credit spread need not increase with maturity if the correlation is strongly negative.

Implications for the Minimum Risk Funding Strategy

A commonly held assumption among corporate treasurers and bank management is that the "zero risk" funding strategy is for the maturity of the liability issued to be matched in a maturity sense to the maturity of the asset being financed. In the context of our model, what does "zero risk" mean? We take the zero risk funding strategy to be the funding strategy that eliminates short-term interest rate volatility from the equity return. This leaves the part of asset volatility that is uncorrelated with interest rates unhedged. Using Ito's lemma, and forcing the asset and debt sensitivities to interest rate changes to be identical in dollar terms, we find the following implicit relation is satisfied at the minimum risk point:

$$B = \frac{[1 - N(h_1)]\,\sigma_V\,\rho k V}{P\,\sigma_r\,N(h_2)(1 - e^{-k\tau})}$$

That is to say, there exist combinations of face value and maturity for debt that eliminate interest rate risk for equity holders.

This formula shows that the minimum risk funding strategy depends on the correlation between "credit risk" and the risk free interest rate, as well as other parameters of the model. In general, that means that a strategy of matching maturities will in general NOT produce the minimum risk funding strategy.

5. Using the Risky Debt Formula for Capital Allocation and Capital Adequacy

The Bankers Trust Company has long used a formula labeled "risk adjusted return on capital" to judge the internal profitability of diverse banking activities on a risk-equalized basis. This concept is explained at length in Uyemura and van Deventer [1992]. In brief, each unit i within the organization conducts business which result in cash flows with a standard deviation of S_i dollars per year. "Capital adequacy" is measured by determining the amount of capital necessary to assure that the unit on a stand-alone basis has a 99 percent probability of remaining solvent. Returns on the business activity are measured by calculating returns related to this risk-adjusted capital figure. The Bank for International Settlements capital adequacy regulations, as implemented by the Federal Reserve Board, are intended to achieve the same objective although the regulations are set at arbitrary risk-weight levels.

How can the risky debt formula above be used for capital allocation? There are three steps in the process:

oSelect the time frame for the analysis (say one year)

oChoose B, the face amount of zero coupon debt in the model, and F so that the continuous yield on cash proceeds of F to earn B in one year is the same as the bank's marginal cost of funds for a one year horizon

oSelect the volatility of interest rates

oDetermine the underlying volatility of the asset class and its correlation with movements in the risk-free interest rate

oSolve for V, the value of assets

The amount of capital that would be allocated to this asset class would be (V-F)/V as a ratio to the value of assets. All asset classes would have the same marginal cost of debt with a one year maturity, and in this sense, the capital ratios are "risk adjusted" and properly consider credit risk, interest rate risk, and the correlation between them.

6. Summary and Suggestions for Future Research

This paper has shown the impact of interest rate movements and "credit risk," broadly defined, on the pricing of risky debt. The correlation between interest rate movements and the returns on the underlying asset is shown to be an important variable in determining the credit spread on risky debt. The formula has broad implications for financial strategy, optimal maturities in debt financing, and capital adequacy analysis and capital allocation.

This analysis can be extended in various ways. First of all, the analysis of a banking firm's risky debt valuation should not employ the lognormality assumption. The value of F given above is not lognormally distributed; a complete analysis of risky debt for banks would assume lognormality of asset returns at the borrower level and use the formula for F as the value of assets V to derive bank debt pricing. Secondly, as pointed out by Ingersoll, one cannot conclude that the value of coupon-bearing debt is simply the sum of risky zero coupon bonds, since each maturity of a zero coupon bond changes the value of assets V. Nonetheless, the valuation formula for risky debt when rates are stochastic represents an important step forward in understanding the full complexity of credit analysis.

Appendix

1. Derivation of the Partial Differential Equation for the Pricing of Risky Debt Formula in the Vasicek Bond Pricing Model

Let F be the value of the risky bond with maturity τ so F = F(V, r, B, τ). V is the value of underlying firm assets, and r is the instantaneous risk-free interest rate. Using the stochastic processes for r and V and applying Ito's lemma gives the following expression for the change in the value of risky debt:

$$
\begin{aligned}
dF &= F_V\,dV + F_r\,dr + F_t\,dt + \frac{1}{2}F_{VV}(dV)^2 + \frac{1}{2}F_{rr}(dr)^2 + F_{rV}(drdV) \\
&= [\alpha V F_V + k(\gamma - r)F_r + F_t + \frac{1}{2}F_{VV}V^2\sigma_V^2 + \frac{1}{2}F_{rr}\sigma_r^2 + F_{rV}\rho\sigma_r\sigma_V V]dt \\
&\quad + [F_V\sigma_V V]\,dz_1 \\
&\quad + [F_r\sigma_r]\,dz_2
\end{aligned}
$$

We then impose no arbitrage conditions by selecting a portfolio such that interest rate risk and asset risk (credit risk in the case of a bank) are eliminated by taking positions in the underlying asset and the risk-free (except for interest rate risk) bond. Assume the portfolio includes one unit of the risky bond and w_1 and w_2 units of the asset and riskless bond respectively.

The value of the portfolio is Q = F $+w_1V$ $+w_2P(\tau)$ where the riskless bond has the same maturity as the risky debt. We choose $w_1 = -F_V$. Now $dQ = dF + w_1dV + w_2dP$. To eliminate interest rate risk, choose $w_2 = -F_r/P_r$. Once the portfolio has been made riskless, the instantaneous return on the portfolio must equal the risk-free instantaneous interest rate; dQ must equal $rQdt = r[F - F_VV - F_r/P_r\, P(\tau)]$, which means that

$$dQ = F_t\, dt + \frac{1}{2} F_{VV} V^2 \sigma_V^2 + \frac{1}{2} F_{rr} \sigma_r^2 + F_{rV}\, \rho\, \sigma_r \sigma_V V - \frac{F_r}{P_r}\left[- P_\tau + \frac{1}{2} P_{rr} (dr)^2 \right]$$
$$= r\left[F - F_V V - \frac{F_r}{P_r} P(\tau) \right]$$

$$dQ = F_t + \frac{1}{2} F_{VV} V^2 \sigma_V^2 + \frac{1}{2} F_{rr} (\sigma_r)^2 + F_{rV}\, \rho\, \sigma_r \sigma_V V + \frac{F_r}{P_r} P_\tau - \frac{F_r}{P_r}\left[\frac{1}{2} P_{rr} \sigma_r^2 \right]$$
$$= r\left[F - F_V V - \frac{F_r}{P_r} P \right]$$

So the fundamental pricing equation is

$$F_t + \frac{1}{2} F_{VV} V^2 \sigma_V^2 + \frac{1}{2} F_{Vr} \sigma_V^2 + F_{rV}\, \rho\, \sigma_r \sigma_V V + \frac{F_r}{P_r} P_\tau - \frac{F_r}{P_r} \frac{1}{2} P_{rr} \sigma_r^2 - rF + rF_v V + r\frac{F_r}{P_r} P = 0$$

To avoid riskless arbitrage in the riskless bond market, the expected return less the risk free rate divided by the bond's volatility has to equal a constant risk aversion parameter lambda. From the partial differential equation for no-arbitrage equilibrium in the riskless bond market, we know

$$\frac{P_r(T)k(\gamma - r) + \frac{1}{2}P_{rr}(T)\sigma_r^2 + P_t(T) - rP(T)}{P_r(T)} = \lambda$$

We substitute this expression into the equation above to obtain the fundamental partial differential equation for the pricing of risky debt:

$$F_t + \frac{1}{2}F_{VV}V^2\sigma_V^2 + \frac{1}{2}F_{rr}\sigma_V^2 + F_{rV}\rho\sigma_r\sigma_V V + F_r[k(\gamma - r) - \lambda] - rF + rF_v V = 0$$

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Referee's Appendix

Verification that Risky Debt Formula Satisfies Partial Differential Equation

Now we show by direct calculation that

$$F = V - M$$
$$= V - V\ N(h_1^*) + BP_{(\tau)} N(h_2^*)$$

(precise definitions of these terms will be given below)

is really a solution to the partial differential equation

$$F_t + \frac{1}{2} F_{VV} V^2 \sigma_V^2 + \frac{1}{2} F_{rr} \sigma_r^2 + F_{rV}\ p\, \sigma_r \sigma_V V + F_r \left[k(\gamma - r) - \lambda\right] - rF + rF_V V = 0,$$
$$\lambda = \frac{P_r k(\gamma - r) + \frac{1}{2}\sigma_r^2 P_{rr} + P_t - rP}{P_r}$$
$$F(\tau = 0) = \min(B,V).$$

The terms appearing in the expression for F are the following,

$$P(\tau) = \exp\left[\frac{1 - e^{-K\tau}}{K}(R(\infty) - r) - \tau R(\infty) - \frac{\sigma_r}{4K^3}(1 - E^{-K\tau})^2\right]$$
$$where\ \ R(\infty) = \gamma + \frac{\sigma_r}{K} q - \frac{1}{2}\frac{\sigma_r^2}{K^2}$$

The instantaneous variance is given by

$$V(s)^2 = \sigma_V^2 + \delta^2 - 2p\,\sigma_V\,\delta \quad where\ \delta = \delta_{(\tau)} = -\frac{1-e^{-K\tau}}{K}\sigma_r,$$

$$we\ set\quad T = \int_0^{\tau} V(s)^2\, ds.$$

Now h_1^* and h_2^* are defined to be:

$$h_1^* = \frac{\log\frac{V}{P_{(\tau)}B} + \frac{1}{2}T}{\sqrt{T}}, \quad h_2^* = \frac{\log\frac{V}{P_{(\tau)}B} - \frac{1}{2}T}{\sqrt{T}}$$

$$N(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} d'x$$

is the normal distribution function.

We set

$$U(\tau) = \frac{1-e^{-k\tau}}{k}$$

$$n(x) = \frac{d}{dx}N(x)$$

for later convenience.

First of all, we calculate the necessary partial derivatives, denoting the Merton formula for an option when rates are stochastic as $M(V,B,\tau)$.

$$M_V = N(h_1^*) + Vn(h_1^*)\,h_{1V}^* - BPn(h_2^*)\,h_{2V}^*$$

$$M_{VV} = 2n(h_1^*)\,h_{1V}^* + V(-h_1 n(h_1^*)\,h_{1V}^{*2} + n(h_1^*)\,h_{1VV}^*) - BP(-h_2 n(h_2^*)\,h_{2V}^{*2} + n(h_2^*)\,h_{2VV}^*)$$

$$M_r = Vn(h_1^*)\,h_{1r}^* - BPn(h_2^*)\,h_{2r}^* - BN(h_2^*)\,P_r$$

$$M_{rr} = V(-h_1^* n(h_1^*)\,h_{1r}^{*2} + n(h_1^*)\,h_{1rr}^*) - BP(-h_2^* n(h_2^*)\,h_{2r}^{*2} + n(h_2^*)\,h_{2rr}^*)$$
$$- 2Bn(h_2^*)\,h_{2r}^*\,P_r - BN(h_2^*)\,P_{rr}$$

$$M_{rV} = n(h_1^*)\,h_{1r}^* + V(-h_1^* n(h_1^*)\,h_{1V}^*\,h_{1r}^* + n(h_1^*)\,h_{1Vr}^*) + BP(-h_2^* n(h_2^*)\,h_{2V}^*\,h_{2r}^*$$
$$+ n(h_2^*)\,h_{2Vr}^*) - Bn(h_2^*)\,h_{2V}^*\,P_r$$

$$M_t = Vn(h_1^*)\,h_{1t}^* - BPn(h_2^*)\,h_{2t}^* - BP_t\,N(h_2^*)$$

Substituting these, the left hand side of the partial differential equation becomes the following. (Note i-th line corresponds to the i-th term of the P.D.E.)

$$-Vn(h_1^*)\,h_{1t}^* + BPn(h_2^*)\,h_{2t}^* + BP_t\,N(h_2^*)$$
$$+\frac{1}{2}V^2\sigma_V^2\{-2n(h_1^*)\,h_{1V}^* - V(-h_1^* n(h_1^*)\,h_{1V}^{*2} + n(h_1^*)\,h_{1VV}^*$$
$$+ BP(-h_2^* n(h_2^*)\,h_{2V}^* + n(h_2^*)\,h_{2VV}^*)\}$$
$$+\frac{1}{2}\sigma_r^2\{-V(-h_1^* n(h_1^*)\,h_{1r}^{*2} + n(h_1^* h_{1rr}^*) + BP(-h_2^* n(h_2^*)\,h_{2r}^{*2}$$
$$+ n(h_2^*)\,h_{2rr}^*) + 2Bn(h_2^*)\,h_{2r}^*\,P_r + BN(h_2^*)\,P_{rr}\}$$
$$+ p\,\sigma_r\,\sigma_V\,V\{-n(h_1^*)\,h_{1r}^* - V(-h_1^* n(h_1^*)\,h_{1V}^*\,h_{1r}^* + n(h_1^*)\,h_{1Vr}^*)$$
$$+ BP(-h_2^* n(h_2^*)\,h_{2V}^*\,h_{2r}^* + n(h_2^*)\,h_{2Vr}^*) + B_n(h_2^*)\,h_{2Vr}^*) + Bn(h_2^*)\,h_{2V}^*\,P_r\}$$
$$-\lambda\{-Vn(h_1^*)\,h_{1r}^* + BPn(h_2^*)\,h_{2r}^* + BN(h_2^*)\,P_r\}$$
$$-r\{V - VN(h_1^*) + BPN(h_2^*)\}$$
$$+ rV\{1 - N(h_1^*) - V_n(h_1^*)\,h_{1V}^*\,BPn(h_2^*)\,h_{2V}^*\}$$

To calculate this, we need the partial derivatives of h_1^* and h_2^*.

$$h_1^* = \frac{\log\frac{V}{PB} + \frac{1}{2}T}{\sqrt{T}}$$

and noting T is independent of r and V,

We have

$$h_{i\,r}^* = -\frac{1}{\sqrt{T}}\frac{P_r}{P}$$

$$h_{i\,r}^{*2} = \frac{1}{T}\frac{P_r^2}{P^2}$$

$$h_{i\,rr}^* = -\frac{1}{\sqrt{T}}\left(\frac{P_{rr}}{P} - \frac{P_r^2}{P^2}\right)$$

$$h_{i\,V}^* = \frac{1}{\sqrt{T}}\frac{1}{V}$$

$$h_{i\,V}^{*2} = \frac{1}{T}\frac{1}{V^2}$$

$$h_{i\,VV}^* = -\frac{1}{\sqrt{T}}\frac{1}{V^2}$$

$$h_{i\,rV}^* = 0 \quad (\ for\ i = 1 \wedge 2\ ),$$

$$h_{1t}^* = -\frac{P_t}{P}\frac{1}{\sqrt{T}} - \frac{1}{2}\frac{T_t}{T^{\frac{3}{2}}}\log\frac{V}{PB} + \frac{1}{4}\frac{T_t}{T^{\frac{1}{2}}}$$

$$h_{2t}^* = -\frac{P_t}{P}\frac{1}{\sqrt{T}} - \frac{1}{2}\frac{T_t}{T^{\frac{3}{2}}}\log\frac{V}{PB} - \frac{1}{4}\frac{T_t}{T^{\frac{1}{2}}}$$

We also see from the definition of P that

$P_r = -U(\tau)P = -UP$ (recall $U(\tau) = \{1 - e^{-k\tau}\}/k$)

$P_r^2 = U^2P^2$,

Finally

$$T_t = \_T_\tau = -V^2(s)$$
$$= -(\sigma_V^2 + \delta^2 - 2p\,\sigma_V\,\delta)$$
$$= -(\sigma_V^2 + U^2\sigma_r^2 + 2p\,\sigma_r\,\sigma_V\,U)$$

Taking all these into account, we can see that the above

expression of the left hand side of the partial differential

equation is zero.

Finally we will see the behavior of F when τ approaches zero.

From the definition of $P(\tau)$ and $T = T(\tau)$, we see immediately

$$P(\tau) \to 1 \quad (\tau \to 0)$$
$$T(\tau) \to +0 \quad (\tau \to 0)$$

So,

$$h_1^* \to +\infty \quad V > B \;\; (\tau \to 0)$$
$$0 \quad V = B$$
$$-\infty \quad V < B \quad for \;\; i = 1,2$$

As N(+∞) = 1, N(0) = 1/2, N(-∞) = 0,

$$F \to B \quad V > B$$
$$V = B \quad V = B \;\; (\tau \to 0), this\ is,$$
$$V \quad V < B$$
$$F(\tau = 0) = \min(V, B).$$

This shows that the boundary condition is satisfied.