

# Forward Rate Curve Smoothing

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## Abstract

This paper reviews the forward rate curve smoothing literature. The key contribution of this review is to link the static curve fitting exercise to the dynamic and arbitrage-free models of the term structure of interest rates. As such, this review introduces more economics to an almost exclusively mathematical exercise, and it identifies new areas for research related to forward rate curve smoothing.

Key words. Forward rate curves, polynomial splines, smoothed splines, term structure evolutions, the HJM model.

JEL Classification. G12, E43

## 1 Introduction

For pricing and hedging fixed income securities, including interest rate derivatives, or for the determination of monetary policy, knowing the current forward rate curve is critical. It is important for pricing and hedging fixed income securities because it is a basic input to the valuation methodology (see Heath, Jarrow, Morton (1992)). With respect to monetary policy, the current forward rate curve is an important source for deducing the market's expectations regarding future spot rates and inflation. As noted, these two applications are fundamentally dynamic in nature.

In contrast, the current literature on forward rate curve smoothing is analyzed almost exclusively in a static setting<sup>1</sup>. For reviews of the standard methodologies see BIS (2005), Hagan and West (2006), and van Deventer, Imai, and Mesler (2013). In the static setting, forward rate smoothing is generally viewed as just a mathematical exercise in curve fitting, with little or no economics involved.

The purpose of this paper is to provide an updated review of the forward rate curve smoothing methodology, in the context of a dynamic setting. The dynamic setting introduces additional economics into the curve fitting exercise.

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<sup>1</sup>To my knowledge, there are no exceptions to this statement.

More importantly, it adds additional structure useful for constructing more valid forward rate curves. In essence, the key contribution of this review is to relate the static forward rate curve smoothing literature to the arbitrage-free term structure evolution literature.

These forward rate smoothing procedures and arbitrage-free term structure models can be applied to any term structure of asset prices; examples include default-free bonds, credit risky bonds, and commodity futures prices. Nonetheless, for pedagogical reasons, this paper focuses only on default-free bonds. The same insights, however, apply more generally to other term structures. In addition, a host of other considerations arise when considering the simultaneous smoothing of multiple term structures of asset prices. Unfortunately, these considerations are delegated to outside reading. Finally, this review focuses on the economics of the forward rate curve smoothing problem. It does not emphasize the empirical evidence nor related computational issues. Although these omitted topics are important, they are left to the existing empirical and applied mathematics literature.

An outline of this paper is as follows. Section 2 discusses forward rate smoothing in the classical static setting. Section 3 extends this analysis to a dynamic context, and section 4 concludes.

## 2 The Static Problem

Fix a time  $t = 0$ . Given are a collection of traded default-free zero-coupon and coupon bond *market* prices. In the U.S. these correspond to Treasury bill, note, and bond prices. These securities have different maturities. For short-term bonds, the time interval between maturities is weekly up to about a year, then spaced at six month intervals thereafter. This price data consists of a finite collection of different maturity observations. *The problem is to infer from these prices the underlying forward rate curve* - a function whose domain is a closed interval of the real line, starting at time  $0$  and ending with the maturity of the longest maturity bond, denoted  $\tau$  (approximately 30 years for U.S. Treasuries).

A forward rate curve whose domain is an interval of the real line,  $[0, \tau]$ , is needed for pricing interest rate derivatives with cash flows at future time points that do not match the maturity dates of the traded bonds, or for forecasting future interest rates over time horizons that do not match the the bond's maturity dates as well. The problem is complicated because one needs to select the "correct" curve, conceptually an uncountable infinite number of points, using only a finite number of observed points to make this determination.

The existing approaches to solving this problem concentrate on a static setting using only the structure known from a single date. We discuss these approaches in this section.

## 2.1 Implied Zero-Coupon Bond Prices

Given is a collection of traded zero-coupon and coupon bond *market* prices at time 0. From these coupon bond prices one must first compute the implied zero-coupon bond prices underlying their values.

To do this inference, we assume that all the relevant bonds (zero-coupon and coupon) are traded at time 0 in frictionless and competitive markets. By frictionless we mean that there are no transaction costs, no restrictions on trade (e.g short sale restrictions), and no differential taxes on capital gains and short term income. By competitive we mean that traders act as price takers. In addition, we assume that these markets admit no arbitrage opportunities. That is, it is impossible to buy and sell different bonds to create a zero cash flow at time 0, nonnegative cash flows for all future times and states, and strictly positive cash flows for some time and states with strictly positive probability.

To do the analysis we need some notation. Let the market price of a zero-coupon bond paying a sure dollar at time  $T$  be denoted  $p(T) > 0$  for  $T \in \{1, \dots, \tau\}$ . The strict positivity is a no arbitrage restriction. We assume that a non-empty subset of these zero-coupon bonds trades.

Let there be  $N$  coupon bonds trading in the economy. Consider the  $j^{\text{th}}$  coupon bond with maturity  $T_j$ , coupon payments  $C_j$  paid every time period up to and including on the maturity date, and a principal of  $L_j$ . Let the price of this coupon bond be denoted  $B_j$  for  $j = 1, \dots, N$ .

If all of the zero-coupon bonds trade as well, then no-arbitrage implies the following expression holds.

$$B_j = \sum_{t=1}^{T_j} C_j p(t) + L_j p(T_j) \text{ for } j = 1, \dots, N. \quad (1)$$

Otherwise one can buy/sell the coupon bond and sell/buy the portfolio of zero-coupon bonds on the right side to obtain an arbitrage opportunity.

Order the maturities of the coupon bonds from smallest to largest, i.e.  $T_1 \leq T_2 \leq \dots \leq T_N$ . Without loss of generality, let the longest maturity coupon bond have maturity  $T_N = \tau$ . Note that although  $N < \tau$  is possible if the traded coupon bonds have missing maturities, all the zero-coupon bonds with maturities from 1, ...,  $\tau$  are implicitly reflected in this set of coupon bond prices.

In matrix form, we can rewrite this expression as:

$$\begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix}_{N \times 1} = \begin{pmatrix} C_1 & C_1 & \cdots & C_1 + L_1 & 0 & 0 & 0 \\ C_2 & C_2 & \cdots & C_2 & C_2 + L_2 & 0 & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \vdots \\ C_N & C_N & \cdots & C_N & \cdots & & C_N + L_N \end{pmatrix}_{N \times \tau} \begin{pmatrix} p(1) \\ \vdots \\ p(\tau) \end{pmatrix}_{\tau \times 1} \quad (2)$$

**Problem 1** (Computing Implied Zero-Coupon Bond Prices). Given the prices of the coupon bonds in the vector on the left side of expression (2) and the bond's

characteristics, as reflected in the matrix on the right side of this expression, to solve for the vector of zero-coupon bond prices.

It is easy to see that a solution exists to this problem if and only if the column vector on the left side of this expression is in the span of the columns of the matrix on the right side of this expression.

For the subsequent analysis we assume such a solution exists and it is unique. In practice, it is always possible to modify the collection of bonds used in the computation so that a unique solution exists. Given this solution, we now have a set of zero-coupon bond prices  $p(T)$  at time 0 for maturities  $T = 1, \dots, \tau$ .

To this collection of implied zero-coupon bond prices we add the prices of the traded zero-coupon bonds. The implied zero-coupon bond prices from solving expression (2) may be different from some (or all) of the traded zero-coupon bond prices. If they exist, these differences are due to observation error, market frictions, or maybe even arbitrage opportunities. Consequently, we need to allow certain maturities to have multiple prices, i.e. for a fixed  $T$ , we denote the collection of zero-coupon bond prices by  $\{p_j(T) : j = 1, \dots, M_T\}$ . This set of zero-coupon bond prices,

$$\{p_j(T) : j = 1, \dots, M_T; T = 1, \dots, \tau\}, \quad (3)$$

is the input to the forward rate curve smoothing procedure discussed in the next section.

## 2.2 Forward Rate Curve Construction

This section discusses how to fit a forward rate curve to the set of zero-coupon bond prices in expression (3). Let  $P : [0, \tau] \rightarrow [0, \infty)$  denote the theoretical zero-coupon bond price function. Note that the domain is now  $[0, \tau]$  and not  $\{0, 1, \dots, \tau\}$ .

Define the  $T$ -maturity forward rate  $f : [0, \tau] \rightarrow \mathbb{R}$  by<sup>2</sup>

$$f(T) = -\frac{\partial \log P(T)}{\partial T}. \quad (4)$$

We assume the zero-coupon price function is such that this derivative exists for all  $T$ . Of course, this implies that

$$P(T) = e^{-\int_0^T f(s) ds}. \quad (5)$$

Note that we use a capital  $P$  for the *theoretical* zero-coupon price function, and a small font  $p$  for the “observed” market prices.

For the purposes of curve fitting, we assume that  $f \in C^k(\mathbb{R})$ , i.e.  $f$  is continuously differentiable up to the  $k$ th order. If  $k = 0$ , then  $f$  is just continuous. The economic interpretation of this assumption is based on recognizing that

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<sup>2</sup>We let the range be the entire real line, allowing for the existence of negative forward rates. These have been observed in recent times.

the forward rate is that rate which one can contract at time 0 on riskless investing beginning at time  $T$  for the time period  $[T, T + dt]$ . It seems unlikely that two “close” dates in the future would not have “close” forward rates. This implies that  $f \in C^0(\mathbb{R})$ . Similar logic can be applied to argue that  $f \in C^k(\mathbb{R})$  for the largest order of  $k$  possible, and that the curve should have “maximum smoothness” as defined in section 2.5.2 below.

The forward rate curve construction problem is to find the forward rate curve that “best” matches the given zero-coupon bond prices in expression (3). More formally,

**Problem 2** (Forward Rate Curve Fitting). Find a function  $f \in C^k(\mathbb{R})$  such that

$$\sum_{T=1}^{\tau} \sum_{j=1}^{M_T} [P(T) - p_j(T)]^2 = \sum_{T=1}^{\tau} \sum_{j=1}^{M_T} [e^{-\int_0^T f(s) ds} - p_j(T)]^2 \quad (6)$$

is minimized.

Here “best” is defined with respect to the  $L^2$  norm. This norm is used for analytic convenience. In this minimization one could use a weighted sum of squared errors if one believed that some of the market prices are more reliable than others.

We point out that the minimization problem in expression (6) is to find the forward rate curve and not the zero-coupon bond price curve. This is done because the forward rate curve is unrestricted, whereas the zero-coupon bond price curve will normally be a decreasing function of maturity  $T$ . This occurs, of course, if forward rates are non-negative. It is easier to solve the unconstrained optimization problem in terms of forward rates, then a constrained optimization problem in terms of zero-coupon bond prices.

Early papers smoothing the zero-coupon bond price curve instead of the forward rate curve include McCulloch (1971), Vasicek and Fong (1982), Shea (1985), and Barzanti and Corradi (1998).

## 2.3 Indeterminacy

To solve problem 2 and to understand the indeterminacy issue, we break the problem up into two steps.

**Step 1:** Remove the multiplicities in the zero-coupon bond prices with identical maturities.

**Problem 3** (Removing Multiple Prices). Find  $(p_1, \dots, p_{\tau}) \in \mathbb{R}^{\tau}$  such that  $\sum_{T=1}^{\tau} \sum_{j=1}^{M_T} [p_T - p_j(T)]^2$  is minimized.

Using standard calculus, the solution to this problem is easily obtained as:

$$p_T^* = \frac{\sum_{j=1}^{M_T} p_j(T)}{M_T} \text{ for } T = 1, \dots, \tau. \quad (7)$$

*Proof.* To get the stationary points, note that

$$\frac{\partial \sum_{T=1}^{\tau} \left( \sum_{j=1}^{M_T} [p_T - p_j(T)]^2 \right)}{\partial p_T} = 2 \sum_{j=1}^{M_T} [p_T - p_j(T)] = 2M_T p_T - 2 \sum_{j=1}^{M_T} p_j(T) =$$

0. The solution is expression (7). Since the objective function is convex, this yields a global minimum.  $\square$

**Step 2:** Given the solution to problem 3, find the forward rate curve by solving a finite set of equations.

**Problem 4** (Forward Rate Curve Fitting). Find

$$\{f \in C^k(\mathbb{R}) : p_T^* = e^{-\int_0^T f(s)ds} \text{ for } T = 1, \dots, \tau\}. \quad (8)$$

We note that the set of solutions is a convex set in  $C^k(\mathbb{R})$ . The dimension of this set is the dimension of the affine subspace in  $C^k(\mathbb{R})$  that contains the convex set. This set is infinite dimensional, i.e. there are an infinite number of linearly independent functions in  $C^k(\mathbb{R})$  that solve this problem. Indeed, to obtain such a solution, first choose any continuously differentiable and strictly positive zero-coupon bond price curve  $P(T) \in C^1(\mathbb{R})$  that contains all the points  $(p_1^*, \dots, p_\tau^*)$ . Then, use the definition of the forward rate, expression (4), to obtain the forward rate curve solution to problem 4.

Consequently, without additional restrictions, the solution to this forward rate curve fitting problem is *indeterminate*. To obtain a unique solution to this forward rate curve fitting problem, the key issue to be addressed is how to remove this indeterminacy. This is the topic discussed in the next section.

## 2.4 Removing the Indeterminacy

To remove this indeterminacy, we need to restrict the solution set to be in a finite dimensional linear subspace of  $C^k(\mathbb{R})$ . With this goal in mind, let's restrict the feasible solutions to an  $n \leq \tau$  - dimensional linear subspace of  $C^k(\mathbb{R})$ , denoted by  $\mathcal{C}_n \subset C^k(\mathbb{R})$ . Since this linear subspace is finite dimensional, there exist  $n$  linearly independent functions  $f_i \in C^k(\mathbb{R})$  for  $i = 1, \dots, n$  such that

$$\mathcal{C}_n = \text{span}\{f_1, \dots, f_n\}. \quad (9)$$

To increase flexibility in the curve fitting exercise, we let these functions depend on a set of  $m$  parameters  $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$ . With this dependence, we rewrite the linear subspace as

$$\mathcal{C}_n(\beta) = \text{span}\{f_1(\beta), \dots, f_n(\beta)\}. \quad (10)$$

Problem 4 can now be rewritten as:

**Problem 5** (Forward Rate Curve Fitting). Choose  $f(\beta) \in \mathcal{C}_n(\beta)$  and  $\beta \in \mathbb{R}^m$  such that

$$\sum_{T=1}^{\tau} \sum_{j=1}^{M_T} [e^{-\int_0^T f(s;\beta)ds} - p_j(T)]^2 \quad (11)$$

is minimized.

By definition of a basis, for an arbitrary  $f(\beta) \in \mathcal{C}_n(\beta)$  there exists an  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$  such that  $f(\beta) = \sum_{j=1}^n \alpha_j f_j(\beta)$ . Therefore, problem 5 is equivalent to the following problem in  $\mathbb{R}^{n+m}$ .

**Problem 6** (Forward Rate Curve Fitting). Choose  $\alpha \in \mathbb{R}^n$  and  $\beta \in \mathbb{R}^m$  such that

$$\sum_{T=1}^{\tau} \sum_{j=1}^{M_T} [e^{-\sum_{j=1}^n \alpha_j \int_0^T f_j(s;\beta) ds} - p_j(T)]^2 \quad (12)$$

is minimized.

In general, this problem needs to be solved numerically.

An important economic issue related to the solution to problem 6 is whether or not the smoothed forward rate curve solution obtained should generate a zero-coupon bond price curve that contains all of the points  $(p_1^*, \dots, p_\tau^*)$  from the solution to problem 3.

The answer to this question depends on one's views concerning the quality of the price data in expression (3). If one believes that each of the market prices for the different maturity zero-coupon bond prices are "good," i.e. just contain observational error, then all of the market prices should be included in the estimation of  $(p_1^*, \dots, p_\tau^*)$ , and  $(p_1^*, \dots, p_\tau^*)$  should be contained on the theoretical zero-coupon bond price curve. If not, then the answer is "no." Of course, the violation of this condition depends on any mispricings embedded in the market prices, and this will depend on the particular market considered.

In the case that this restriction is desired to be imposed on the forward rate curve smoothing solution, then in general, one can only satisfy this constraint if the space of basis functions  $\mathcal{C}_n(\beta)$  satisfies the following spanning property.

**Spanning Property:** The solution subspace  $\mathcal{C}_n(\beta)$  is such that given any  $(p_1^*, \dots, p_\tau^*) \in \mathbb{R}^\tau$ , there exists a  $\beta \in \mathbb{R}^m$  such that

$$\{f \in \mathcal{C}_n(\beta) : p_T^* = e^{-\int_0^T f(s;\beta) ds} \text{ for } T = 1, \dots, \tau\} \neq \emptyset. \quad (13)$$

Not all sets of basis functions  $\mathcal{C}_n(\beta)$  will satisfy this property. However, if  $\mathcal{C}_n(\beta)$  satisfies this spanning property, then the two-step procedure used in solution to problems 3 and 4 above applies. The result is a forward rate curve whose implied zero-coupon bond price curve contains the points  $(p_1^*, \dots, p_\tau^*)$ , and the optimization problem 4 on this basis set  $\mathcal{C}_n(\beta)$  reduces to solving the system of non-linear equations given in expression (14).

**Problem 7** (Forward Rate Curve Fitting). If  $\mathcal{C}_n(\beta)$  satisfies the spanning property, then the solution to problem 6 constrained so that  $P(T)$  contains the solution  $(p_1^*, \dots, p_\tau^*)$  to problem 3 is given by any  $\alpha^* \in \mathbb{R}^n$  and  $\beta^* \in \mathbb{R}^m$  that satisfies

the following matrix equation:

$$\begin{bmatrix} \log(p_1^*) \\ \vdots \\ \log(p_\tau^*) \end{bmatrix}_{\tau \times 1} = \begin{bmatrix} -\int_0^1 f_1(s : \beta^*) ds & \cdots & -\int_0^1 f_n(s : \beta^*) ds \\ \vdots & & \vdots \\ -\int_0^\tau f_1(s : \beta^*) ds & \cdots & -\int_0^\tau f_n(s : \beta^*) ds \end{bmatrix}_{\tau \times n} \begin{bmatrix} \alpha_1^* \\ \vdots \\ \alpha_n^* \end{bmatrix}_{n \times 1}. \quad (14)$$

*Proof.* Note that in problem 6, imposing the constraint, the two step procedure discussed previously for solving problem 2 applies. The second step is to find  $\alpha \in \mathbb{R}^n$  and  $\beta \in \mathbb{R}^m$  such that  $p_T^* = e^{-\sum_{j=1}^n \alpha_j \int_0^T f_j(s:\beta) ds}$  must hold for each  $T$ . Alternatively,  $\log(p_T^*) = -\sum_{j=1}^n \alpha_j \int_0^T f_j(s : \beta) ds$ . Writing this in matrix form completes the proof.  $\square$

In application of this solution technique, if necessary additional constraints can be imposed to guarantee a unique solution. Some examples will be enlightening.

## 2.5 Examples

The choice of the basis functions in

$$\mathcal{C}_n(\beta) = \text{span}\{f_1(\beta), \dots, f_n(\beta)\} \quad (15)$$

determines the forward rate curve solution to problem 5. We now explore various choices used in the industry and the academic literature. The bases utilized are mainly selected for analytic or computational convenience.

### 2.5.1 Exponential-Polynomial Bases

Exponential-polynomial functions are the basis functions employed by most central banks, with the exception of Canada, Japan, the United Kingdom, and the United States (see BIS (2005)). The linear subspace considered is:

$$\mathcal{C}_4(\beta) = \text{span}\{1, e^{\beta_1 T}, T e^{\beta_2 T}, T e^{\beta_3 T}\} \quad (16)$$

where  $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ . It is easy to see that these functions are linearly independent.

This is known as the Svensson (1994) family. A special case is the Nelson-Siegel (1987) family. It is important to note<sup>3</sup> that the exponential-polynomial basis functions do not satisfy the spanning property, and consequently this basis will generate a forward rate curve whose implied zero-coupon bond price curve does not contain all of the “observed” zero-coupon bond prices  $(p_1^*, \dots, p_\tau^*)$  solving problem 3.

<sup>3</sup>Trivially, there will be more zero-coupon bond prices  $T = 1, \dots, \tau$  than there are basis functions. The implied zero-coupon bond price curve is very unlikely to contain many of the observed zero-coupon bond prices, unless additional constraints are imposed in the optimization problem.

## 2.5.2 Polynomial Splines

Polynomial splines are the basis functions employed by Canada, Japan, the United Kingdom, and the United States (see BIS (2005)). The linear subspace considered is obtained by the following construction. First, the time domain  $[0, \tau]$  is partitioned into a collection of subintervals whose union is the entire domain:  $\bigcup_{i=0, \dots, n} [t_i, t_{i+1}] = [0, \tau]$  where  $t_0 = 0 \leq t_1 \leq \dots \leq t_n = \tau$ . The times  $\{t_i : i = 1, \dots, n\}$  are called the “knot points.” The basis functions are then defined as  $d$ th degree polynomials over these subintervals, i.e.

$$f_i(T; \beta) = [\beta_{1i} + \beta_{2i}T + \beta_{3i}T^2 + \dots + \beta_{di}T^d]1_{[t_i, t_{i+1}]}(T) \quad (17)$$

for  $T \in [0, \tau]$  and  $i = 1, \dots, n$  where  $\beta = (\beta_{ji} : j = 1, \dots, d; i = 1, \dots, n) \in \mathbb{R}^m$  for all  $j, i$  with  $m = d + n$ .

Instead of polynomials in the basis functions as in expression (17), alternative functions could be employed. Exponential splines such as  $[\beta_1 + \beta_2 e^{\beta_3 T} + \beta_4 T e^{\beta_5 T}]$  are common, see for example Vasicek and Fong (1982), Shea (1985), Barzanti and Corradi (1998), and Hagan and West (2006).

As given, these basis functions are not in  $C^k(\mathbb{R})$  for  $d \geq k \geq 0$ . To ensure that these functions are in  $C^k(\mathbb{R})$ , one needs to add constraints on the coefficients. These constraints join the separate functions to ensure they are continuously differential to the  $k$ th order. For a  $k$ th degree continuously differential polynomial spline, they are given by:

$$\begin{aligned} C^0(\mathbb{R}) : \quad f_i(t_{i+1}) &= f_{i+1}(t_{i+1}) \\ C^1(\mathbb{R}) : \quad f'_i(t_{i+1}) &= f'_{i+1}(t_{i+1}) \\ &\vdots \\ C^k(\mathbb{R}) : \quad f_i^{(k)}(t_{i+1}) &= f_{i+1}^{(k)}(t_{i+1}) \end{aligned} \quad (18)$$

for all  $i = 1, \dots, n - 2$  where  $(j)$  in  $f_i^{(j)}(T)$  denotes the  $j$ th derivative. The conditions are imposed only on the interior knot points.

Note that the basis functions in expression (17) are linearly independent. In general, polynomial splines will not satisfy the spanning property because the number of knot points can be strictly less than the number of observed zero-coupon bond price maturities, i.e.  $n < \tau$ .

However, the polynomial spline can be made to satisfy the spanning property if the knot points are set equal to the maturity dates  $T = 1, \dots, \tau$  of the zero-coupon bonds in expression (3). In this case, polynomial splines will generate forward rate curves whose implied zero-coupon bond price curve contains all the observed zero-coupon bond prices. This is in contrast to the exponential-polynomial basis functions, where in general this matching is impossible. This is one reason why many economists prefer polynomial splines to using the set of exponential-polynomial basis functions for forward rate smoothing, see van Deventer, Imai, and Mesler (2013).

In selecting a polynomial spline, both the knot points and the properties of the spline at the end points of the domain, the maturities  $T = 0$  and  $\tau$  also need

to be determined. One can arbitrarily choose the values of the curves at these points or their derivatives, or a combination of all of these.

In his thesis, Janosi (2004) gives an interesting economic argument on the selection of the properties for the smoothing function near  $\tau$ . He argues that information available in the market to form expectations regarding events at maturity dates far into the future, say 20 to 30 years, is not that different. Consequently, using the fact that forward rates equal expected future spot rates plus a risk premium (see Jarrow (2009)), he argues that the expectations component as a function of maturity should be very smooth - approximately linear. Combined with slowing changing risk premium as maturity increases, this implies asymptotic linearity in the smoothing function near  $\tau$ .

### Maximum Smoothness Forward Rate Curves

As to the selection of the degree of the polynomial spline, it can be shown (see Adams and van Deventer (1994), van Deventer, Imai, and Mesler (2013)) that a 4th degree polynomial spline with the knot points equal to the zero-coupon bonds' maturity dates  $(1, \dots, \tau)$  is the "smoothest" forward rate curve consistent with matching the observed zero coupon bond prices, i.e. a 4th degree polynomial spline is the solution to:

$$\min_{f(\beta) \in C^3(\mathbb{R})} \left\{ \int_0^\tau [f''(s, \beta)]^2 ds : p_T^* = e^{-\int_0^T f(s; \beta) ds} \text{ for } T = 1, \dots, \tau \right\}. \quad (19)$$

This makes a 4th degree polynomial spline for the forward rate curve construction a justifiable selection. Note that as defined, maximum smoothness forward rate curves satisfy the spanning property.

### Smoothed Splines

When the number of knot points is strictly less than the number of bonds ( $n < \tau$ ) and the spanning property is not imposed on the polynomial spline, an alternative to maximum smoothness forward rate curves is to use smoothed polynomial splines. Such a spline is obtained by employing the polynomial splines from expression (17) and solving the following problem:

**Problem 8** (Forward Rate Curve Fitting). Choose  $\alpha \in \mathbb{R}^n$  and  $\beta \in \mathbb{R}^m$  such that

$$\sum_{T=1}^{\tau} \sum_{j=1}^{M_T} [e^{-\sum_{j=1}^n \alpha_j \int_0^T f_j(s; \beta) ds} - p_j(T)]^2 + \lambda \left( \int_0^\tau [f''(s, \beta)]^2 ds \right) \quad (20)$$

is minimized where  $\lambda > 0$ .

The parameter  $\lambda$  is called the *roughness penalty*. One can think of problem 8 as representing the Lagrangian from a constrained version of problem 5, where the constraint is the smoothness measure  $\int_0^\tau [f''(s, \beta)]^2 ds$ . The choice of  $\lambda$ 's magnitude is arbitrary and part of the problem statement. Its choice can be

included as an extra parameter to be determined in any empirical validation of the smoothing methodology. Examples of studies using smoothed splines include Fisher, Nychka, and Zervos (1995), Waggoner (1997), and Andersen (2007).

## 2.6 Empirical Validation

The economics of the static problem imposes very few constraints on the choice of the basis functions. There are two approaches one can use to clarify the basis function selection.

One can use economic theory to construct an equilibrium model to determine the shape of the term structure. Unfortunately, equilibrium models depend critically on the assumed primitives of the economy (preferences, endowments, beliefs) and the market clearing mechanism. For practical applications, this approach is problematic because one can not guarantee that the equilibrium model's structure matches market realities.

Alternatively, one can use an empirical validation procedure. One approach is to use "out-of-sample" validation. In its simplest form, this consists of removing a subsample of zero-coupon bond maturities from the given data set. Next, fit a forward rate curve to the remaining zero-coupon bond prices. Then, see how close the theoretical zero-coupon bond prices are to the omitted subsample of zero-coupon bond prices. The basis that performs best in the out-of-sample test is the preferred basis.

## 3 The Dynamic Model

This section presents a dynamic model for forward rate smoothing construction. The uses of forward rate smoothing - pricing/hedging derivatives and inferring market expectations of future rates - are in a dynamic context. Hence, it is important to understand the constraints, if any, that dynamic considerations impose on the static forward rate smoothing. This is the purpose of this section.

We consider a discrete time, finite horizon Heath, Jarrow, Morton (1992) model for the evolution of the term structure of interest rates. Discrete time is selected for expositional simplicity. All of the following results are obtainable in a continuous time setting.

Given is a complete filtered probability space  $(\Omega, \mathcal{F}_\tau, \mathbb{F}, \mathbb{P})$  where the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t=0, \dots, \tau}$  satisfies the usual hypotheses. Here  $\mathbb{P}$  is the statistical probability measure. It is assumed that traded in frictionless and competitive markets are the collection of default-free zero-coupon bonds that pay sure dollars at times  $[1, \dots, \tau]$ .

To do the analysis, we need to extend the notation for our zero-coupon bonds and forward rates to include this dynamic element. In particular, the time  $t$  price of a zero-coupon bond maturing at time  $T \geq t$  is denoted  $P(t, T)$ ,

and the forward rate is defined by

$$f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}.$$

It is assumed that the forward rate curve evolves according to the following stochastic process:

$$\Delta f(t, T) = \mu(\omega, t-1, T) + \sum_{j=1}^K \sigma_j(\omega, t-1, T) \Delta W_j(t) \text{ for } t = 1, \dots, \tau \quad (21)$$

$$f(0, T) = f_0(T) \in C^k(\mathbb{R})$$

where  $\Delta g(t) \equiv g(t) - g(t-1)$  for an arbitrary function  $g(t)$ ,  $\mu(\omega, t-1, T)$  and  $\sigma_j(\omega, t-1, T) > 0$  for all  $j = 1, \dots, K$  are  $\mathcal{F}_{t-1}$ -measurable, and  $\{W_j(t) : j = 1, \dots, K\}$  are independent standard Brownian motions adapted to the filtration  $\mathbb{F}$ .

This is a very general stochastic process. As given, the forward rate  $f(t, T)$  evolves stochastically in time with a drift  $\mu(\omega, t-1, T)$  and  $K$  random shocks with respective volatility coefficients  $\sigma_j(\omega, t-1, T)$  for  $j = 1, \dots, K$ . This is called a  $K$ -factor model. For more discussion of such stochastic term structure models see Jarrow (2009).

### 3.1 Arbitrage-Free Evolution

To be consistent with well functioning markets, we only want to consider forward rate curve evolutions that are arbitrage-free. Heath, Jarrow, and Morton (1992)<sup>4</sup> show that a necessary and sufficient condition for the evolution in expression (21) to be arbitrage-free is the existence of a  $K$  vector of  $\mathcal{F}_{t-1}$ -measurable stochastic processes  $(\theta_1(t), \dots, \theta_K(t))$  such that:

$$\mu(\omega, t-1, T) = -\sum_{j=1}^K \sigma_j(\omega, t-1, T) \left[ \theta_j(\omega, t-1) - \sum_{S=t-1}^T \sigma_j(\omega, t-1, S) \right]. \quad (22)$$

This is known as the HJM drift restriction. The stochastic processes  $\{\theta_j(t) : j = 1, \dots, K\}$  in expression (21) represent risk premiums for the interest rate risks generated by the  $K$  Brownian motion processes. Key in this drift restriction for the  $T$ -maturity forward rate is the fact that the risk premiums do not depend on the forward rate's maturity.

### 3.2 Empirical Validation

For the purposes of this paper, it is important to emphasize that there is an enormous literature on the empirical validation and estimation of HJM models (see Dai and Singleton (2003) for a review). In this context, the volatility

<sup>4</sup>See also Jarrow and Turnbull (2000), chapter 16, for this discrete time representation.

functions  $\{\sigma_j(\omega, t, T) : j = 1, \dots, K\}$  can be estimated using historical time series data or calibrated to the market prices of traded interest rate derivatives. In addition, the risk premium  $\{\theta_j(\omega, t) : j = 1, \dots, K\}$  can also be estimated in a similar manner. Consequently, from the perspective of constructing forward rate curves, the volatilities and risk premium can be considered as “known” quantities. This observation will be important in the subsequent analysis.

### 3.3 Consistency

This section studies the constraints imposed on the basis functions  $(\mathcal{C}_n(\beta))$  in the static forward rate construction due to the notion of dynamic consistency. The idea of dynamic consistency is due to Bjork and Christensen (1999). When considering a dynamic evolution of the forward rate curve, one wants to choose the basis functions such that the smoothed forward rate curves constructed in the static setting are those that can be possibly generated in the future by the term structure evolution, expression (21). If this is true, then the forward rate smoothing procedure is called *consistent*. If it is not true, then the evolution and the forward rate curve construction are *inconsistent*.

Inconsistency leads to problems in the two primary uses of the smoothed forward rate curves. The first is in the pricing and hedging of interest rate derivatives. When hedging such a derivative, deltas (sensitivities of the change in the derivative’s price to a change in the forward rate curve) need to be computed. If consistency is violated, the change in the derivative’s value computed will be different from the true change in value because  $\Delta f_t$  will be incorrectly estimated. The reason is because the starting position for the change is wrong.

The second is in using the forward rate curve to infer the market’s expectations of future spot rate realizations. If inconsistent forward rate curves are used, then the current forward rate curve used for the estimation could not have been generated by historical evolutions of the forward rate curve (expression (21)). In addition, curves like it can never be realized in the future. Consequently, the implied expectations based on such a curve are irrational with respect to past experiences and all possible future realizations.

To generate the constraints imposed on the basis functions by consistency, we need to add additional structure on the general forward rate curve evolution given in expression (21).

**Assumption** (Time-to-maturity) The volatilities are a function of only time-to-maturity and a parameter vector  $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$  for all  $j = 1, \dots, K$ , i.e.

$$\sigma_j(\omega, t - 1, T) = \sigma_j(T - t + 1, \beta).$$

This assumption facilitates estimation and forward rate curve smoothing.<sup>5</sup> For

<sup>5</sup>Since the volatility functions are  $\mathcal{F}_{t-1}$ -measurable, this can be generalized to make the volatility functions depend on the state of the economy at time  $t - 1$ , e.g. the level of forward rates.

simplicity of notation, let's define the time-to-maturity as  $\mathcal{T} = T - t + 1$ .

Using this assumption, we can rewrite the forward rate curve evolution as:

$$f(t, \mathcal{T} + t - 1) = f(t - 1, \mathcal{T} + t - 1) + \mu(t - 1, \mathcal{T} + t - 1, \beta) + \sum_{j=1}^K \Delta W_j(t) \sigma_j(\mathcal{T}, \beta). \quad (23)$$

This expression can be used to understand the restrictions necessarily imposed on the spanning functions for the static forward rate curve construction due to the dynamic nature of the forward rate curve's evolution.

Consider problem 2 where we are standing at time  $t$ , and we desire to construct the forward rate function

$$f(t, \mathcal{T} + t - 1) \equiv f_t(\mathcal{T}, \beta) \in C^k(\mathbb{R}).$$

The right side of expression (23) reveals the *only spanning functions* that are consistent with the arbitrage-free evolution of the term structure of interest rates.

1. First we have the previous forward rate curve at time  $t - 1$ , i.e. the<sup>6</sup> function  $f_{t-1}(\mathcal{T}, \beta) \in C^k(\mathbb{R})$ . In general, the forward rate curve depends on  $\beta$  due to it being the solution to expression (21). This function can be arbitrarily specified.
2. Second, on the extreme right side of this expression we have the volatility functions  $\{\sigma_1(\mathcal{T}, \beta), \dots, \sigma_K(\mathcal{T}, \beta)\} \in C^k(\mathbb{R})$ . Without loss of generality we can assume that these functions are linearly independent, otherwise we can redefine the Brownian motions to obtain a notationally simpler evolution with fewer independent Brownian motions. Note that the "constants" preceding these basis functions correspond to random draws from the Brownian motion differences  $\Delta W_j(t)$ .
3. Last, we have the arbitrage-free drift function:

$$\begin{aligned} \mu(t - 1, \mathcal{T} + t - 1, \beta) &= - \sum_{j=1}^K \sigma_j(\mathcal{T}, \beta) \theta_j(t - 1) + \sum_{j=1}^K \sigma_j(\mathcal{T}, \beta) \sum_{S=0}^{\mathcal{T}} \sigma_j(S, \beta) \\ &\equiv \mu_{t-1}(\mathcal{T}, \beta) \in C^k(\mathbb{R}). \end{aligned}$$

Note that at time  $t - 1$ , the risk premiums  $\theta_j(t - 1)$  for all  $j$  are not functions of  $\mathcal{T}$ . Thus, the first term is in  $\text{span}\{\sigma_1(\mathcal{T}, \beta), \dots, \sigma_K(\mathcal{T}, \beta)\}$ . The second term is completely determined by the volatility functions and it is a function of  $\mathcal{T}$ . It can be proven that each individual term in the sum,  $\sigma_j(\mathcal{T}, \beta) \sum_{S=0}^{\mathcal{T}} \sigma_j(S, \beta)$ , are not in the  $\text{span}\{\sigma_1(\mathcal{T}, \beta), \dots, \sigma_K(\mathcal{T}, \beta)\}$ , which implies that the set of functions  $\{\mu_{t-1}(\mathcal{T}, \beta), \sigma_1(\mathcal{T}, \beta), \dots, \sigma_K(\mathcal{T}, \beta)\}$  are linearly independent.

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<sup>6</sup>We point out that starting at time 0 the initial forward rate curve can be arbitrarily selected since it is exogenously specified in expression (21).

*Proof.* For simplicity, we omit  $\beta$  from the notation.

First, consider the two vectors  $\sigma_j(\mathcal{T}) \sum_{S=0}^{\mathcal{T}} \sigma_j(S)$  and  $\sigma_j(\mathcal{T})$ .

Suppose there exists constants such that

$$a_0 \sigma_j(\mathcal{T}) \sum_{S=0}^{\mathcal{T}} \sigma_j(S) + a_1 \sigma_j(\mathcal{T}) = \sigma_j(\mathcal{T}) \left[ a_0 \sum_{S=0}^{\mathcal{T}} \sigma_j(S) + a_1 \right] = 0.$$

This implies that  $\sum_{S=0}^{\mathcal{T}} \sigma_j(S) = -\frac{a_1}{a_0}$  for all  $\mathcal{T}$ .

In general, this can only be true if  $a_0 = a_1 = 0$ . Hence, these vectors are linearly independent.

Next, consider the two vectors  $\sigma_j(\mathcal{T}) \sum_{S=0}^{\mathcal{T}} \sigma_j(S)$  and  $\sigma_k(\mathcal{T})$  for  $j \neq k$ .

Suppose there exists constants, nonzero, such that

$$a_0 \sigma_j(\mathcal{T}) \sum_{S=0}^{\mathcal{T}} \sigma_j(S) + a_1 \sigma_k(\mathcal{T}) = 0.$$

This implies  $\frac{\sigma_j(\mathcal{T})}{\sigma_k(\mathcal{T})} \sum_{S=0}^{\mathcal{T}} \sigma_j(S) = -\frac{a_1}{a_0}$  for all  $\mathcal{T}$ .

In general, this can only be true if  $a_0 = a_1 = 0$ .

Hence, these vectors are linearly independent. This completes the proof.  $\square$

In summary, combining these three observations, the spanning functions on the right side of expression (23) are

$$\{f_{t-1}(\mathcal{T}, \beta), \mu_{t-1}(\mathcal{T}, \beta), \sigma_1(\mathcal{T}, \beta), \dots, \sigma_K(\mathcal{T}, \beta)\}.$$

This implies that the relevant linear subspace of  $C^k(\mathbb{R})$  for the construction of the static forward rate curve at time  $t$  must lie in the linear subspace

$$\mathcal{C}_n(\beta) \equiv \text{span}\{f_{t-1}(\mathcal{T}, \beta), \mu_{t-1}(\mathcal{T}, \beta), \sigma_1(\mathcal{T}, \beta), \dots, \sigma_K(\mathcal{T}, \beta)\}.$$

For a fixed  $\beta$ , this set has a dimension of at least  $K + 1$ . Its dimension is  $K + 2$  depending upon whether or not  $f_{t-1}(\mathcal{T}, \beta)$  is in the span of the remaining functions. We discuss issues related to the selection of the basis function  $f_{t-1}(\mathcal{T}, \beta)$  in a subsequent section.

### 3.4 Examples of Inconsistent Smoothing Bases

In the static model, using specific functional forms for the forward rate curve's basis determined without knowledge of the term structure's evolution often generates a consistency problem. This section provides two such examples for commonly used models. For additional examples of inconsistent bases and forward rate evolutions in the continuous time setting see Bjork and Christensen (1999) and Filipovic (2001).

#### 3.4.1 The Ho and Lee Model

This example shows that the Ho and Lee model is inconsistent with the Svensson forward rate curve smoothing basis. The Ho and Lee model is a special case of the evolution in expression (21) where there is only a single Brownian motion and the volatility function satisfies:

$$\sigma(T - t + 1) = \sigma > 0$$

Changing the notation to the time-to-maturity, i.e.  $\mathcal{T} = T - t + 1$ , yields the forward rate evolution

$$f(t, \mathcal{T} + t - 1) = f(t - 1, \mathcal{T} + t - 1) + \mu_{t-1}(\mathcal{T}, \sigma) + \Delta W(t)\sigma. \quad (24)$$

Direct substitution into expression (21) gives the arbitrage-free drift:

$$\mu_{t-1}(\mathcal{T}, \sigma) = -\sigma\theta(t - 1) - \sigma^2 \times \mathcal{T}. \quad (25)$$

We see that the drift is in the  $\text{span}\{1, \mathcal{T}\}$ . Combining expressions (24) and (25) we see that the spanning set for the forward rate curve at time  $t$  is:

$$\{f_{t-1}, 1, \mathcal{T}\}.$$

Now, let's consider the Svensson basis, which is

$$\{1, e^{\beta_1 T}, T e^{\beta_2 T}, T e^{\beta_3 T}\}.$$

Note that  $\mathcal{T}$  is linearly independent of this basis. Thus,  $f_t$  is not in the span of this basis, and the Svensson basis is inconsistent with the Ho and Lee model. As shown, the problem arises due to the arbitrage-free drift term, and not the Brownian motion shocks.

### 3.4.2 The Extended Vasicek Model

This example shows that the extended Vasicek model is inconsistent with the Svensson forward rate curve smoothing basis. The extended Vasicek model is a special case of the evolution in expression (21) where there is only a single Brownian motion and the volatility function satisfies:

$$\sigma(T - t + 1; \sigma, \kappa) = \sigma e^{-\kappa(T-t+1)} > 0$$

where  $\sigma > 0, \kappa > 0$ . This example is easily extended to multiple factors with different volatility functions.

Changing the notation to time-to-maturity we have:

$$f(t, \mathcal{T} + t - 1) = f(t - 1, \mathcal{T} + t - 1) + \mu_{t-1}(\mathcal{T}; \sigma, \kappa) + \Delta W(t) \cdot \sigma e^{-\kappa \mathcal{T}} \quad (26)$$

Direct substitution into expression (21) gives the arbitrage-free drift:

$$\mu_{t-1}(\mathcal{T}; \sigma, \kappa) = -\sigma e^{-\kappa \mathcal{T}} \left[ \theta(t - 1) - \sum_{S=0}^{\mathcal{T}} \sigma e^{-\kappa S} \right]. \quad (27)$$

Combining expressions (26) and (27) we see that the spanning set is:

$$\{f_{t-1}, e^{-\kappa \mathcal{T}}, e^{-\kappa \mathcal{T}} \sum_{S=0}^{\mathcal{T}} \sigma e^{-\kappa S}, 1\}.$$

Now, let's consider the Svensson basis, which is

$$\{1, e^{\beta_1 T}, T e^{\beta_2 T}, T e^{\beta_3 T}\}.$$

Even setting  $\beta_1 = \beta_2 = -\kappa$ , it can be shown that  $e^{-\kappa T} \sum_{S=0}^T \sigma e^{-\kappa S}$  is linearly independent of this basis. Thus,  $f_t$  is not in the span of this basis, and the Svensson basis is inconsistent with the extended Vasicek model. As shown, the problem arises due to the arbitrage-free drift term, and not the Brownian motion shocks.

### 3.5 Empirical Implementation

As pointed out in section 3.3, the dynamic term structure model leaves “one degree of freedom” in the selection of the basis functions for use in the static forward rate smoothing construction. This is the selection of the function  $f_{t-1}$ . All of the other basis functions are determined (known) from the drift and volatility estimates of the term structure's evolution, see section 3.2. The purpose of this section is to discuss methods for selecting  $f_{t-1}$ .

First, an empirical approach can be utilized. Specific functional forms can be hypothesized based on analytic convenience and then empirically validated or rejected. This approach is similar to that previously used in the static forward rate curve fitting methodology. Pursuing this determination is a fruitful area for future research.

Alternatively, one can assume that  $f_{t-1}$  is in the span of the basis functions generated by the drifts and volatilities from the term structure evolution, i.e.  $f_{t-1} \in \mathcal{C}_n(\beta) = \text{span}\{\mu_{t-1}(\beta), \sigma_1(\beta), \dots, \sigma_K(\beta)\}$ . The justification for the assumption is based on a “steady state” argument. Looking at the forward rate curve's evolution in expression (21), we can deduce that  $f_{t-1}$  is a function of the initial forward rate curve  $f_0$ , and past forward rate drifts and volatility functions. In a stationary steady state economy, perhaps the influence of the initial forward rate curve on the current forward rate curve is minimal and it can be ignored. Hence, the assumption.

## 4 Conclusions

The key contribution of this review is to link the static forward rate curve fitting exercise to the dynamics of the forward rate curve's stochastic evolution. This linkage is missing in the literature, and it generates new economic insights about the static curve fitting exercise. In particular, it is shown that the basis set of functions used in a static setting to fit a forward rate curve is almost completely determined by the forward rate curve evolution's drift and volatility functions. It is conjectured that these new insights should generate improved forward rate curve estimates when applied in practice. The verification of this conjecture awaits future research.

## References

- [1] Adams, K., van Deventer, D. 1994. Fitting yield curves and forward rate curves with maximum smoothness. *Journal of Fixed Income*. June: 52-62
- [2] Andersen, L. 2007. Discount curve construction with tension splines. *Review of Derivatives Research*. 10: 227 - 267
- [3] Barzanti, L., Corradi, C. 1998. A note on interest rate term structure estimation using tension splines. *Insurance: Mathematics and Economics*. 22: 139 - 143.
- [4] Bank for International Settlements. 2005. Zero-coupon yield curves: technical document. BIS Papers.
- [5] Bjork, T., Christensen, B. 1999. Interest rate dynamics and consistent forward rate curves. *Mathematical Finance*. 9 (4): 323 - 348
- [6] Dai, Q., Singleton, K. 2003. Term structure dynamics in theory and reality. *Review of Financial Studies*. 16 (3): 631 - 678
- [7] Filipovic, D. 2001. *Consistency problems for Heath Jarrow Morton interest rate models*. Lecture Notes in Mathematics. Berlin: Springer.
- [8] Fisher, M., Nychka, D., Zervos, D. 1995. Fitting the term structure of interest rates with smoothing splines. Working paper 95 -1, Finance and Economics Discussion Series, Federal Reserve Board.
- [9] Janosi, T. 2004. *Arbitrage-free forward rate curves and applications*. Dissertation Cornell University.
- [10] Jarrow, R. 2009. The term structure of interest rates. *Annual Review of Financial Economics*. 1:69-96.
- [11] Jarrow, R., Turnbull, S. 2000. *Derivative Securities*. 2nd edition. South-Western College Publishing.
- [12] Hagan, P., West, G. 2006. Interpolation methods for curve construction. *Applied Mathematical Finance*. 13 (2): 89 - 129
- [13] Heath, D., Jarrow, R., Morton, A. 1992. Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica*. 60: 77-105
- [14] McCulloch, J. 1971. Measuring the term structure of interest rates. *Journal of Business*. 44: 19 - 31
- [15] Nelson, C., Siegel, A. 1987. Parsimonious modeling of yield curves. *Journal of Business*. 60 (4): 473 - 489
- [16] Shea, G. 1985. Interest rate term structure estimation with exponential splines: a note. *Journal of Finance*. 40: 319 - 325

- [17] Svensson, L. 1994. Estimating and interpreting forward interest rates: Sweden 1992 - 1994. CEPR Discussion paper 1051 (October).
- [18] van Deventer, D., Imai, K., Mesler, M. 2013. *Advanced Financial Risk Management: tools and techniques for integrated credit risk and interest rate risk management*. 2nd edition. John Wiley & Sons: Singapore.
- [19] Vasicek, O., Fong, H.G. 1982. Term structure modeling using exponential splines. 37: 339 - 348
- [20] Waggoner, D. 1997. Spline methods for extracting interest rate curves from coupon bond prices. Working paper 97-10, Federal Reserve Bank of Atlanta.