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Contingent Claim Valuation with a Random Evolution of Interest Rates*

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Abstract

This paper describes a new approach for pricing contingent claims given a random evolution of interest rates. The methodology incorporates multiple factors and nonnegative interest rates, and it does not require estimates of the "market prices" for risk. As such, it is analogous to the Black-Scholes approach for pricing equity options. To illustrate this methodology, some preliminary estimates of the required parameters are provided and these are utilized to price the Chicago Board of Trade's T-bond futures and T-bond futures options.

1. Introduction

This paper describes a new approach for pricing contingent claims given a random evolution of interest rates. This approach is different from that of Cox, Ingersoll, Ross [*3*] or Brennan and Schwartz [*2*] in that we do not need to explicitly model such quantities as the "market prices for risk" or the expected rates of returns on zero-coupon bonds to price contingent claims. Rather, our method utilizes the concept of a "risk-neutral valuation operator" or a "martingale measure" as described in the seminal papers by Harrison and Kreps [*5*] and Harrison and Pliska [*6*], to eliminate these quantities from the valuation formulas. This method is already well-known to practitioners; in fact, it is analogous to that used in the binomial option pricing model (for equities) based on the calculation of "pseudo" probabilities (see Cox and Rubinstein [*4*]).

Consequently, our approach to pricing contingent claims only requires an explicit modeling of the volatilities for the zero-coupon bond price processes. As such, this approach is analogous to that used by Black-Scholes in pricing equity options. Recall that the Black-Scholes formula does not require estimates of expected returns to compute values, but only volatilities. In fact, it is this characteristic of the Black-Scholes formula that partly accounts for its widespread application in practice. Estimating expected returns (or equivalently the market price for risk) is a difficult task since they are stochastic and nonstationary.

The origin of our model can be traced back to an earlier paper by Ho and Lee [*10*]. Ho and Lee's model takes the initial zero-coupon bond price curve as a given. Second, they let the entire zero-coupon bond price curve fluctuate randomly across time in an arbitrage-free manner, according to a single discrete time binomial process. From this process, they then price contingent claims.

Our approach extends and generalizes the Ho and Lee model in three significant ways. First, we generalize their work to include multiple random factors, so that default-free bonds of different maturities can have positive, but not perfectly correlated, returns. In Ho and Lee's model, all zero-coupon bond returns are perfectly correlated. Second, we extend their model to incorporate continuous trading. From a practical perspective, this allows one to parameterize the model differently (and more simply) than in the Ho and Lee [*10*] case so that the "pseudo" probabilities required in their valuation formula need not be estimated. Third, forward rates, instead of bond prices, are used as the basic building blocks for the analysis. This change in perspective is justified by the stationarity of the forward rate process, especially in cases where the bond price process is not. This stationarity, of course, facilitates the estimation of the parameters for the stochastic processes involved.

The purpose of this paper is twofold. First, it describes the valuation methodology of Heath, Jarrow, Morton [8] in a simpler, more intuitive, and less abstract way. Second, it describes a numerical procedure for computing HJM values using a multinomial tree. Preliminary (and nonexhaustive) calculations are provided to demonstrate both the feasibility of computing contingent claim values with this approach and their accuracy. These calculations are performed for the Chicago Board of Trade's Treasury bond futures and Treasury bond futures options.

An outline of this paper is as follows. Section 2 presents the model structure, Section 3 presents a deterministic economy, Section 4 a one-factor model, and Section 5 a two-factor model. Section 6 analyzes the issues involved in estimating volatilities. Section 7 provides preliminary estimates of the model's values for CBOT Treasury bond futures prices and Treasury bond futures options. Finally, Section 8 concludes the paper.

2. The Model Structure

We are interested in pricing arbitrarily specified contingent claims on the term structure of interest rates. Consequently, we need to provide enough notation and structure so that we can completely describe the stochastic evolution for all zero-coupon bond prices.

Let $P(t,T)$ be the time t price of a zero-coupon bond paying \$1 at date T. The trading dates (t) and maturities (T) go from date 0 to date τ. We assume that all bond prices are strictly positive (i.e., $P(t,T) > 0$).

Let $f(t,T)$ be the instantaneous *forward rate* at time T as seen from date t. It corresponds to the rate one could contract for at time t on a riskless loan over the forward period $[T, T + dT]$. The rate one could contract for at time t on a riskless loan over the next instant, $[t, t + dt]$, is known as the *spot rate* and denoted by $r(t) = f(t,t)$. This correspondence between spot rates and forward rates indicates why the forward rate $f(t,T)$ is often interpreted as the "market's forecast" for the spot rate to prevail at date T, $r(T)$.

The relationship between bond prices and forward rates is well-known and given by

$$P(t, T) = \exp\left\{-\int_t^T f(t, u)du\right\}. \tag{1}$$

Consequently, to describe the random evolution of the term structure, we could characterize either bond prices ($P(t,T)$) or forward rates ($f(t,T)$. Given one, the evolution of the other is then determined by expression (1).

In the sequel, we shall describe some specific models for the evolution of forward rates. When specifying the evolution of these quantities, one is specifying the evolution of the prices of traded securities. As such, it is important to ensure that these processes are consistent with some equilibrium. This is done by ensuring that these processes don't admit arbitrage opportunities. An *arbitrage opportunity* is any self-financing trading strategy involving the underlying bonds that turns a zero initial investment into an amount that is nonnegative (for sure) and that is greater than zero with positive probability. It would be foolish, indeed, to price contingent claims based on a term-structure model that admits such opportunities. Such a model would be internally inconsistent as it would price contingent claims assuming no arbitrage opportunities exist, yet they would exist within the model.

The substance of the remaining analysis consists of (1) guaranteeing that the exogenous process specified for forward rates or bond prices is arbitrage-free, and (2) showing how to price contingent claims given the process. To demonstrate the issues involved in the simplest case, and to motivate which process (bonds or forward rates) to characterize, we first study a model with a deterministic evolution of the term structure.

3. A Deterministic Model of the Term Structure

When the evolution of the term structure is deterministic, all future bond prices ($P(t,T)$) and all future forward rates ($f(t,T)$) are known at date 0. This is the meaning of the word *deterministic*. In such an economy, the absence of arbitrage implies that each of the following conditions must be true:

$$P(s, T) = P(s, t)P(t, T) \quad \text{for } 0 \leq s < t < T \leq \tau. \tag{2.a}$$

and

$$f(s, T) = f(t, T) \quad \text{for } 0 \leq s \leq t \leq T \leq \tau. \tag{2.b}$$

Condition (2.a) has the following interpretation. The left side of (2.a) represents the price paid at date s to get a dollar at time T by investing in a T maturity bond. The right side of (2.a) represents another way at date s to guarantee a certain dollar at time T. First invest in a t maturity bond, hold it until it matures, then roll over the proceeds into the T maturity bond. The quantity $P(s,t)P(t,T)$ represents the price of this second investment strategy. Unless $P(s,T)$ equals $P(s,t)P(t,T)$, an arbitrage opportunity results.

Condition (2.b) also has a nice interpretation. The left side represents the rate one could contract for at *time* s on a riskless loan over $[T, T + dT]$. The right side represents the rate one could contract for at *date* t on a riskless loan over $[T, T + dT]$. If these two rates differed, one could go into the business of borrowing and lending funds for $[T, T + dT]$ with a zero initial investment, but generating strictly positive returns at time T, i.e., an arbitrage opportunity.

As one may have guessed, it is easy to show that expressions (2.a) and (2.b) are in fact equivalent. Expression (2.b), involving forward rates, however, is seen to be the simpler of the two. Furthermore, in the deterministic economy, forward rates are seen to be constant across time while bond prices are not. This simplicity of the evolution of forward rates makes them the preferred quantity to isolate for the subsequent analysis.

Given the no-arbitrage forward rate structure as in expression (2.b), it is easy to understand the pricing of zero-coupon bonds and contingent claims on the term structure. First, from (2.b), we have that $r(u) = f(t,u)$ for all $t \leq u$. Combined with expression (1), this yields

$$P(t, T) = \exp\left\{-\int_t^T r(u)du\right\}. \tag{3}$$

The zero-coupon bond price at time t is seen to be the discounted value of $1 received at date T, where the discount rate is the instantaneous spot rate. This is the deterministic equivalent of the *expectations hypothesis*, i.e., that all bonds earn the same instantaneous return over all periods.

Next, to price a contingent claim with a cash flow of C_T at time T, its time t price is:

$$C_t = C_T \exp\left\{-\int_t^T r(u)du\right\}. \tag{4}$$

For example, if the contingent claim is a European option on the discount bond $P(t, \tau)$ with exercise price K and maturity T, then $C_T =$ $\max[P(T, \tau) - K, 0]$ and $C_t = \max[P(T, \tau) - K, 0] \exp\left\{-\int_t^T r(u)du\right\}$.

Expression (4) provides the valuation operator for computing contingent claim values.

Last, for comparison with the subsequent analysis, we rewrite expression (4) in one additional way. To do this, let

$$B(t) \equiv \exp\left\{\int_0^t r(u)du\right\} \tag{5}$$

represent the time t value of a dollar invested in a *money market account* at time 0, which grows at the spot rate. With this quantity, expression (5) is equivalent to

$$\frac{C_t}{B(t)} = \frac{C_T}{B(T)}. \tag{6}$$

The left side of expression (6) represents the value of the contingent claim at date t in units of the money market account. The right side of expression (6) represents the contingent claim's value at date T also in units of the money market account. These ratios are seen to be equal and constant across time. Adding randomness to the term structure complicates the analysis, but the essential concepts remain unchanged. We now turn to this analysis.

4. A Single-Factor Model of the Term Structure

This section provides a discrete time approximation to a single-factor version of HJM. Alternatively, it provides an example of a discrete trading economy in which contingent claims are valued under a random evolution of the term structure. As a special case of this analysis, one can also obtain the model of Ho and Lee [*10*].

For analysis, we divide the trading horizon $[0,\tau]$ into equal periods of length h. Motivated by the previous section, let us concentrate on modeling the stochastic process for forward rates. We start out with an initial forward rate curve $\{f(0,T):T \in [0,\tau]\}$, which is chosen to match the observed forward rate curve at time 0. Next, for each time t (some integer multiple of h), let us suppose that the time $t + h$ forward rate is given by

$$f(t + h, T) + \begin{cases} f(t, T) + \sigma(t, T, f(t, T))\sqrt{h} \\ \quad \text{with probability } q_t(h) = ½ + o(h) \\ f(t, T) - \sigma(t, T, f(t, T))\sqrt{h} \\ \quad \text{with probability } 1 - q_t(h) \end{cases} \tag{7}$$

where $1 > q_t(h) > 0$, $\lim_{h \to 0} \frac{o(h)}{h} = 0$, and $\sigma(t, T, f(t, T))$ is a function dependent upon time, the maturity of the forward rate, and the forward rate itself.[1] For the discrete time economy this function is unrestricted, but for the continuous time approximation (as $h \to 0$) it must be bounded in some way to avoid forward rates exploding with positive probability, see HJM [*8*]. The dynamics in expression (7) are very simple. Forward rates follow a binomial process that goes up by $\sigma(t,T,f(t,T))\sqrt{h}$ with probability $q_t(h)$ or down by $\sigma(t,T,f(t,T))\sqrt{h}$ with probability $1 - q_t(h)$.

The expectation and variance of the change in forward rates are given by:

$$E_t(f(t + h, T) - f(t, T)) = o(h) \tag{8.a}$$

and

$$\text{Var}_t(f(t + h, T) - f(t, T)) = \sigma^2(t, T, f(t, T))h + o(h) \tag{8.b}$$

where $E_t(\cdot)$ is the expectation operator at time t based on the probability $q_t(h)$ and $\text{Var}_t(\cdot)$ is the variance operator at time t based on the probability $q_t(h)$.

Thus, the expected change in forward rates is almost zero (for small h) and the function $\sigma(t,T,f(t,T))$ has the interpretation of being the forward rate's *volatility* per unit time (for small h). Unfortunately, as written, the process for forward rates given in expression (7) can admit arbitrage opportunities as the following example shows. This example is closely related to one studied by

[1]This particular form for the function $\sigma(t, T, f(t, T))$ was chosen to facilitate estimation. It could easily be generalized to depend on the entire past history of all the forward rates.

Boyle [1].

Example

Let the length of the trading interval $h \equiv 1$. Suppose that the initial forward rate curve $f(0,T)$ is flat and equal to 0.10 for all T. (If time is measured in years, this is 10 percent per annum.) Further, let $f(1,T)$ be given by:

$$f(1,T) = \begin{cases} .12 \text{ with probability } q > 0 \\ \text{or} \\ .08 \text{ with probability } 1 - q > 0 \end{cases}$$

Under this rate structure, we can generate arbitrage profits as follows. Define z by:

$$z = \frac{\exp(-.08) - \exp(-.12)}{\exp(-.16) - \exp(-.24)}.$$

At time 0, buy

$$n_1 = \exp(-.1) - \exp(-.2)\, z \text{ bonds expiring at time 1,}$$

sell

$$n_2 = 1 \text{ bond(s) expiring at time 2,}$$

and buy

$$n_3 = z \text{ bonds expiring at time 3.}$$

The value of this portfolio at time 0 is:

$$\exp(-.1)n_1 - \exp(-2)n_2 + \exp(-.3)n_3, \text{ which is easily seen to be 0.}$$

At time 1, there are two possibilities for the value, one for each possible rate structure. If rates rise, the value is:

$$V_{rise} = \exp(-.1) - \exp(-.2)z - \exp(-.12) + \exp(-.24)z$$

while if rates fall, it is

$$V_{fall} = \exp(-.1) - \exp(-.2)z - \exp(-.08) + \exp(-.16)z.$$

But z was chosen exactly so that these would be equal. Their common value is approximately 0.000181, a small (but sure) positive amount. Hence, this is an arbitrage opportunity! Q.E.D.

Although this very simple and plausible model allows arbitrage, it is interesting to note that it is possible to modify it very slightly (i.e., change the values of $f(1,T)$ a little bit, depending on T) to make arbitrage impossible. Since it is just as easy to describe the modification for the general process as for the example, we do it for the general case.

The idea is to adjust the process in expression (7) by a small quantity, $\delta(t,T)$, to remove the arbitrage opportunity. The process, with the adjustment, is

$$f(t+h,T) = \begin{cases} f(t,T) + \sigma(t,T,f(t,T))\sqrt{h} + \delta(t,T)h \\ \quad \text{with probability } q_t(h) \\ f(t,T) - \sigma(t,T,f(t,T))\sqrt{h} + \delta(t,T)h \\ \quad \text{with probability } 1 - q_t(h) \end{cases} \tag{9}$$

Its expected value and variance are:

$$E_t(f(t+h,T) - f(t,T)) = \delta(t,T)h + o(h) \text{ and}$$

$$\text{Var}_t(f(t+h,T) - f(t,T)) = \sigma^2(t,T,f(t,T))h + o(h), \text{ respectively.}$$

The adjustment leaves the variance unchanged but changes the expected forward rates over the period.

Consider the dynamics in expression (9), but with the probability $q_t(h)$ replaced by ½. Denote expectations with respect to these new probabilities, called *pseudo* probabilities, by $\tilde{E}_t(\bullet)$. We claim that the process in (9) is arbitrage-free if $\delta(t,T)$ is chosen to make

$$\tilde{E}_t(P(t,t+h)P(t+h,T)) = P(t,T). \tag{10}$$

Note the analogy to expression (2.a) in the preceding section.[2]

The appendix shows that the solution is:

$$\delta(t,T)h = \frac{\partial}{\partial T}\ln\left(\cosh\left(\int_{t+h}^{T}\sigma(t,u,f(t,u))\sqrt{h}\,du\right)\right) \tag{11}$$

[2]The condition that $\tilde{E}_t(P(t,s)P(s,T)) = P(t,T)$ for $s > t + h$ is, in fact, not true in general due to the correlation between the forward rates over $[t+h,s]$ and those in $[s,T]$. To see this note that from expression (13),

$$P(t,T) = \tilde{E}_t\left(\frac{P(s,T)}{B(s)}\right)B(t) \text{ and } \tilde{E}_t\left(\frac{1}{B(s)}\right)B(t) = P(t,s), \text{ so}$$

$$P(t,T) = P(t,s)\tilde{E}_t(P(s,T)) + \widetilde{\text{cov}}_t\left(\frac{1}{B(s)}, P(s,T)\right)B(t)$$

$$= \tilde{E}_t(P(t,s)P(s,T)) + \widetilde{\text{cov}}_t\left(\frac{1}{B(s)}, P(s,T)\right)B(t).$$

where $\cosh(x) = (e^{x} + e^{-x})/2$. For the example,[3] the adjustment $\delta(0,T)$ to the expected change in the process for $[f(1,T) - f(0,T)]$ is .02 tanh (.02(T − 1)), which for $T = 1$ is 0, for $T = 2$ is .0004, and for $T = 3$ is .0008. These are small, but nonzero adjustments when T exceeds 2.

To prove the claim that $\delta(t,T)$ in expression (11) makes the dynamics in (9) arbitrage-free, we first observe that in the discrete trading economy, the money market account's value is:

$$B(t) = [P(0,h)P(h,2h) \cdots P(t-h,t)]^{-1}$$

since trading can only take place at intervals of length h. From expression (10) and the law of iterated expectations, we have:

$$\begin{aligned} P(t,T) &= \tilde{E}_t(P(t,t+h)\,\tilde{E}_{t+h}(P(t+h,t+2h)\,P(t+2h,T))) \\ &= \tilde{E}_t(P(t,t+h)\,P(t+h,t+2h)\,P(t+2h,T)) \\ &\vdots \\ &= \tilde{E}_t(P(t,t+h)\,P(t+h,t+2h)\cdots P(T-h,T)\,P(T,T)) \\ &= \tilde{E}_t(P(T,T)/B(T))B(t). \end{aligned}$$

This implies

$$P(t,T) = \tilde{E}_t(P(T,T)/B(T))\,B(t). \tag{12}$$

This says that under the adjustment $(\delta(t,T))$ and with respect to the pseudo probabilities $\tilde{E}_t(\bullet)$, the expectations hypothesis holds. That is, taking expectations with respect to $\tilde{E}_t(\bullet)$ and discounting at the risk-free rate (as determined by the money market account) gives the current price of the bond. Hence, all bonds earn the same expected return under $\tilde{E}_t(\bullet)$. Consequently, expression (12) provides a "risk-neutral valuation" operator.

Alternatively, rewriting expression (12) as

$$\frac{P(t,T)}{B(t)} = \tilde{E}_t\left(\frac{P(T,T)}{B(T)}\right)$$

[3]For the example, the detailed calculations are:

$$\begin{aligned} \delta(0,T) &= \frac{\partial}{\partial T}\left[\ln\left(\cosh\left(\int_1^T (.02)\,du\right)\right)\right] \\ &= \frac{\partial}{\partial T}[\ln(\cosh(.02(T-1)))] \\ &= \frac{.02\sinh(.02(T-1))}{\cosh(.02(T-1))} = .02\tanh(.02(T-1)), \end{aligned}$$

where $\sigma(t,T,f(t,T)) \equiv .02$ and $h = 1$.

shows that under $\tilde{E}_t(\cdot)$, the bond's price relative to the money market account is a *martingale* (hence, the terminology the *martingale measure* technique).

Next in the argument, we claim that the value of a (suitably bounded) self-financing trading strategy relative to the money market account will also be an $\tilde{E}_t(\cdot)$ martingale. This is because portfolio values are linear combinations of the relative bond prices, $\{P(t,T)/B(t)\}$. Being a martingale, if a trading strategy has zero initial investment, it must have a zero expected value under $\tilde{E}_t(\cdot)$. Hence, if a self-financing trading strategy has a positive probability of making money, it must have a positive probability of losing money; for the details, see Heath and Jarrow [7]. No self-financing trading strategy can, therefore, be an arbitrage opportunity under $\tilde{E}_t(\cdot)$. Finally, there can be no arbitrage opportunities under $E_t(\cdot)$ as well since both probabilities agree on zero-probability events. This completes the proof of the claim.

Now (at time t), consider a contingent claim with a payoff of C_T at time T where C_T can depend on, at most, the history of all forward rates up to and including time T. Using an argument similar to that contained in Cox and Rubinstein [4] for equity options, it can be shown that C_T can be obtained at time T through a self-financing trading strategy in the underlying bonds initiated at date t. The market, under the forward rate dynamics of expression (9), is thus said to be *complete*. By the above argument, however, the value of this claim relative to the money market account must be an $\tilde{E}_t(\cdot)$ martingale, which means that

$$C_t = \tilde{E}_t(C_T/B(T))B(t). \tag{13}$$

Expression (13) provides the "risk-neutral valuation operator" analogous to that in the deterministic economy (expression (4)) for pricing contingent claims.

This expectation can be calculated in a recursive fashion starting at date T, and working backward until time t. An example would be for a European bond option on a discount bond $P(t,\tau)$ with maturity $\tau > T$. Let T be the exercise date of the option and K its exercise price. Then,

$$C_T = \max[P(T,\tau) - K, 0], \text{ so}$$

$$C_t = \tilde{E}_t(\max[P(t,\tau) - K, 0]/B(T))B(t).$$

To calculate this value, one starts at time $T - h$ and computes the expectation using expression (13), i.e.,

$$\tilde{E}_{T-h}(\max[P(T,\tau) - K, 0]/B(T)).$$

Next, move back to $T - 2h$, and calculate the expectation of $\tilde{E}_{T-h}(\max[P(T,\tau) - K, 0]/B(T))$. The law of iterated expectations states that this equals

$$\tilde{E}_{T-2h}(\tilde{E}_{T-h}(\max[P(T,\tau) - K, 0]/B(T))) = \tilde{E}_{T-2h}(\max[P(T,\tau) - K, 0]/B(T)$$

One continues in this fashion until time t is reached. This is the "risk-neutral valuation" procedure. This can be described as calculating the expectation by working backward through the "tree" until reaching time t. The same procedure works for American bond options as well, but with the adjustment that, at each step backward in the tree, the decision to exercise early at that step must be considered. If exercise is optimal, the value of the option at that step is replaced by its early exercise value. Otherwise, it iterates as above.

We emphasize that this calculation of contingent claim values involves (from expressions (9) and (13)) only knowledge of (1) the initial forward rate curve $\{f(0,T): T \epsilon [0,\tau]\}$, (2) the volatility function $\sigma(t,T,f(t,T))$, (3) the correction factor $\delta(t,T)$, and (4) the details of the contract. But, the correction factor itself depends only on the volatility function, $\sigma(t,T,f(t,T))$ (see expression (11)). Hence, contingent claim values depend only on the initial forward rate curve and the volatilities. More importantly, they do not depend on the "market price for risk" or the parameters of the expected return on bonds. This is analogous to that independence of expected returns obtained by the Black-Scholes formula when pricing equity options.

Before proceeding to a two-factor model, we note that if $\sigma(t,T,f(t,T))$ is taken to be a positive constant σ, independent of [t, T, and $f(t,T)$], then the process in (9) corresponds to that of Ho and Lee [*10*]. For a formal proof of this statement, see HJM [*9*]. As $h \rightarrow 0$, it can also be shown that the Ho and Lee process converges to

$$f(t,T) = f(0,T) + \sigma^2(Tt - t^2/2) + \sigma W(t) \tag{14}$$

where $\{W(t): t \epsilon [0,\tau]\}$ is a standard Wiener process (see HJM [*9*] or Morton [*11*]). Under this model, closed-form solutions for European bond options are available.

5. A Two-Factor Model of the Term Structure

A single-factor model for the evolution of the term structure is useful for clarifying the economic and mathematical concepts involved; however, it is not as useful for applications. Indeed, it implies that all zero-coupon bonds of different maturities are perfectly correlated, and this is counter to the empirical evidence. To incorporate less than perfect correlation across zero-coupon bonds, we need to utilize at least a two-factor model. Fortunately, having seen the analysis for the single-factor case, the extension is straightforward.

The two-factor model starts with an initial forward rate curve $\{f(0,T): T \epsilon [0,\tau]\}$ that is chosen to match the observed forward rate curve at date 0, and involves a dynamic process that proceeds from step t to $t + h$ as follows:

$$f(t+h,T) = \begin{cases} f(t,T) + \sigma_1(t,T,f(t,T))\sqrt{h} + \delta(t,T)h \\ \quad \text{with probability } q_t(h) = 1/2 + o(h) \\ f(t,T) - \sigma_1(t,T,f(t,T))\sqrt{h} \\ \quad + \sqrt{2}\sigma_2(t,T,f(t,T))\sqrt{h} + \delta(t,T)h \\ \quad \text{with probability } p_t(h) = 1/4 + o(h) \\ f(t,T) - \sigma_1(t,T,f(t,T))\sqrt{h} \\ \quad - \sqrt{2}\sigma_2(t,T,f(t,T))\sqrt{h} + \delta(t,T)h \\ \quad \text{with probability } 1 - q_t(h) - p_t(h) \end{cases} \tag{15}$$

where $1 > q_t(h) > 0$, $1 > p_t(h) > 0$, $\sigma_1(t,T,f(t,T))$ and $\sigma_2(t,T,f(t,T))$ are volatility functions (suitably bounded), and $\delta(t,T)$ is the adjustment term needed to make the process arbitrage-free.

The process in expression (15) differs from the one-factor case, expression (9), by the introduction of one additional branch in the tree. This is, in fact, the minimal number of branches needed to get the process to converge (as $h \to 0$) to a continuous time forward rate process involving two independent Brownian motions, see HJM [8]. The choice of limits for the probabilities $(q_t(h), p_t(h), 1 - q_t(h) - p_t(h))$ as (½, ¼, ¼) is arbitrary to the extent that we could change both expression (15) and (½, ¼, ¼) as long as the mean and variance of the process as in expression (16) below remain unchanged.

$$E_t(f(t+h,T) - f(t,T)) = \delta(t,T)h + o(h) \tag{16}$$

and

$$\mathrm{Var}_t(f(t+h,T) - f(t,T)) = \sigma_1(t,T,f(t,T))^2h + \sigma_2(t,T,f(t,T))^2h + o(h).$$

For a practical example, consider setting the volatilities equal to:

$$\sigma_1(t,T,f(t,T)) = \sigma_1(T-t)\min(1,f(t,T))$$

and (17)

$$\sigma_2(t,T,f(t,T)) = \sigma_2(T-t)\min(1,f(t,T))$$

where $\sigma_1(T-t)$ and $\sigma_2(T-t)$ are functions of the time remaining until maturity. This yields a forward rate process with nonnegative interest rates. This stochastic process has two random shocks; the first is a long-run "shock," roughly proportional to $f(t,T)$ with proportionality factor $\sigma_1(T-t)$, and the second "shock" represents a spread between short and long rates roughly proportional to $f(t,T)$, but with the volatility $\sigma_2(T-t)$. This process possesses many of the properties desired when approximating the observed forward rate dynamics. For a specific parameterized example, one could set $\sigma_1(T - t) \equiv \sigma_1$ and $\sigma_2(T - t) \equiv \sigma_2 e^{-\lambda T}$ for constants $\sigma_1, \sigma_2, \lambda$. This example is discussed at greater length in HJM [8].

By analogy with the one-factor case, the adjustment term $\delta(t,T)$ is selected so that the expectations hypothesis holds under a different set of pseudo probabilities, i.e.,

$$\tilde{E}_t(P(t,t + h)P(t + h,T)) = P(t,T)$$

where $\tilde{E}_t(\cdot)$ is expectation based on the probabilities (½, ¼, ¼) instead of $(q_t(h), p_t(h), 1 - q_t(h) - p_t(h))$.

Given the stochastic process as in (15) and these pseudo probabilities, computing contingent claim values is an easy exercise in calculating expectations. Analogous to the one-factor case, define the money market account value as:

$$B(t) = [P(0,h)P(h,2h) \cdots P(t - h,t)]^{-1}.$$

Let C_T be the cash flow to a contingent claim at time T. A contingent claim dependent on the term structure is any financial security whose cash flows can be duplicated (in a dynamic fashion) with a trading strategy involving the zero-coupon bonds alone. Under the above forward rate dynamics, the market is complete; hence one can duplicate all contingent claims on the term structure.

By an analogous argument to that used in the one-factor case, the time t value of the contingent claim relative to the money market account is a martingale under $\tilde{E}_t(\cdot)$ and, therefore, the contingent claim's value is:

$$C_t = \tilde{E}_t(C_T/B(T))B(t)$$

where the expectation is obtained using expression (15) with "$q_t(h)$" replaced by "½" and "$p_t(h)$" replaced by "¼." These values can now be calculated in a recursive fashion from the multinomial tree implied by expression (15).

The procedure for generalizing the two-factor model to three or more factors is now apparent. One merely appends additional branches on the tree (as in going from expression (9) to (15)) and repeats the same analysis. Contingent claim valuation is still simply a process of calculating discounted expectations. Of course, the computational burden increases with each additional factor.

6. Estimation of the Volatility Functions

To use the model advocated in Section 5, the volatility functions need to be estimated. Broadly speaking, there are two methods for estimation, either implicit or historic. To use implicit estimation, a parameterized version of the volatility functions like that given in the paragraph below expression (17) is needed. Since this procedure is straightforward and similar to that used in standard option pricing theory, little more will be said about it here. In contrast, we concentrate on the historic estimation procedure.

A historic estimation procedure can be utilized to both identify the number of factors and simultaneously estimate the volatility functions. We illustrate this procedure with a k-factor model. The observations available from historic data are proportional changes in $n + k - 1$ different forward rates for equally spaced observation times $t_1, t_2, \cdots, t_n$ and equally spaced maturity dates $T_1, T_2, \cdots, T_{n+k-1}$. The spacing intervals are the same for both of these parameters. In matrix form, this is:

$$\begin{bmatrix} \Delta f(t_1, T_1)/f(t_1, T_1) & \cdots & \Delta f(t_1, T_{n+k-1})/f(t_1, T_{n+k-1}) \\ \Delta f(t_2, T_1)/f(t_2, T_1) & \cdots & \Delta f(t_2, T_{n+k-1})/f(t_2, T_{n+k-1}) \\ \vdots & & \vdots \\ \Delta f(t_n, T_1)/f(t_n, T_1) & \cdots & \Delta f(t_n, T_{n+k-1})/f(t_n, T_{n+k-1}) \end{bmatrix} \tag{18}$$

where $\Delta f(t_i, T) = f(t_i + 1, T) - f(t_i, T)$ for $i = 1, \cdots, n$.

The n rows in the matrix of expression (18) correspond to n observations of historic proportional changes, while the $n + k - 1$ columns correspond to the $n + k - 1$ different maturity forward rates.

For short time periods between observations, the changes in forward rates (under expressions like (15)) are approximately normally distributed. For the proportional case given in expression (17) (as expanded to incorporate up to k factors), the statistical model for (18) then takes the form

$$\frac{\Delta f(t_i, T)}{f(t_i, T)} - \frac{\delta(t_i, T)\Delta t_i}{f(t_i, T)} \approx \sum_{j=1}^{k} \sigma_j(T - t_i)\,\Delta W_j(t_i)$$

$$\text{for } i = 1, \cdots, n \text{ and } T \in \{T_1, \cdots, T_{n+k-1}\},$$

where $\Delta t_i \equiv t_{i+1} - t_i$ and $\{\Delta W_1(t), \cdots, \Delta W_k(t) : t \in \{t_1, \cdots, t_n\}\}$ are k orthogonal normally distributed random variables. As written, this expression ignores the upper bound on the proportional process as indicated in expression (17).

For statistical analysis, we stagger the observations (discarding some forward rate maturities) such that the columns of the adjusted observation matrix correspond to identical times until maturity. In matrix form, the statistical model becomes

$$\begin{bmatrix} \frac{\Delta f(t_1, T_1)}{f(t_1, T_1)} - \frac{\delta(t_1, T_1)\Delta t_1}{f(t_1, T_1)} & \cdots & \frac{\Delta f(t_1, T_k)}{f(t_1, T_k)} - \frac{\delta(t_1, T_k)\Delta t_1}{f(t_1, T_k)} \\ \frac{\Delta f(t_2, T_2)}{f(t_2, T_2)} - \frac{\delta(t_2, T_2)\Delta t_2}{f(t_2, T_2)} & \cdots & \frac{\Delta f(t_2, T_{k+1})}{f(t_2, T_{k+1})} - \frac{\delta(t_2, T_{k+1})\Delta t_2}{f(t_2, T_{k+1})} \\ \vdots & & \vdots \\ \frac{\Delta f(t_n, T_n)}{f(t_n, T_n)} - \frac{\delta(t_n, T_n)\Delta t_n}{f(t_n, T_n)} & \cdots & \frac{\Delta f(t_n, T_{k+n-1})}{f(t_n, T_{k+n-1})} - \frac{\delta(t_n, T_{k+n-1})\Delta t_n}{f(t_n, T_{k+n-1})} \end{bmatrix} = \tag{19}$$

$$\begin{bmatrix} \Delta W_1(t_1)\ \Delta W_2(t_1) & \cdots & \Delta W_k(t_1) \\ \Delta W_1(t_2)\ \Delta W_2(t_2) & \cdots & \Delta W_k(t_2) \\ \vdots & & \\ \Delta W_1(t_n)\ \Delta W_2(t_n) & \cdots & \Delta W_k(t_n) \end{bmatrix} \begin{bmatrix} \sigma_1(\tau_1)\ \sigma_2(\tau_1) & \cdots & \sigma_k(\tau_1) \\ \sigma_1(\tau_2)\ \sigma_2(\tau_2) & \cdots & \sigma_k(\tau_2) \\ \vdots & & \vdots \\ \sigma_1(\tau_k)\ \sigma_2(\tau_k) & \cdots & \sigma_k(\tau_k) \end{bmatrix}'$$

where $\tau_{i+1} \equiv (T_{j+i} - t_j)$ for $i = 0, \cdots, k-1$ and $j = 1, \cdots, n$ are constants for each i, and the "prime" denotes transpose. The proportional volatility functions are the columns of the matrix

$$\Omega \equiv \begin{bmatrix} \sigma_1(\tau_1) \cdots \sigma_k(\tau_1) \\ \vdots \quad\quad \vdots \\ \sigma_1(\tau_k) \cdots \sigma_k(\tau_k) \end{bmatrix}$$

which are assumed to be constant across time as given in expression (17).

Given expression (19), a *principal components analysis* can be used to determine the volatility matrix Ω, see Theil [*12*]. The latent roots of the principal components analysis $(\lambda_1, \cdots, \lambda_k)$, normalized, will yield the percent of variation explained by the ith factor $(\Delta W_i(t))$. The characteristic vectors $(a_1, \cdots, a_k)$ will give estimates of the proportional volatility functions, i.e.,

$$a_i = \begin{bmatrix} \sigma_i(\tau_1) \\ \vdots \\ \sigma_i(\tau_k) \end{bmatrix} \text{ for } i = 1, \cdots, k.$$

These estimates can be used directly in the computation of contingent claim values. We illustrate the application of these estimates in the next section.

7. Treasury Bond Futures and Treasury Bond Futures Options

This section illustrates the HJM valuation methodology by applying it to price CBOT Treasury futures and Treasury futures options. The computations are not meant to be exhaustive, rather, they serve only to document the feasibility and accuracy of computing with this methodology. The computations were made utilizing a program, written by Heath and

Morton, and distributed by BARRA. The program computes values utilizing the procedure outlined in Sections 4 and 5 above.

To use the model, we must first establish the current forward rate curve. For this simple analysis, we used a step function (a piecewise constant curve) as a first approximation to the forward rate curve. We let it be constant on the intervals: [0,1), [1,3), [3,5), [5,7), [7,10), [10,20), and [20,∞). (Times are measured in years.) We obtained prices of stripped Treasuries maturing near the times 1, 3, 5, 7, 10, 20, and 30 from the November 10 issue of *The Wall Street Journal*. These prices are given in Table 1. All prices in this section are expressed in dollars and cents.

We computed constant forward rates over the time intervals ending and beginning at the date of these instruments (using the average of the bid and ask prices),[4] and then (since these times were not exactly the times desired) computed (weighted) averages of these rates over the desired intervals. The resulting forward rates used for analysis are given in Table 2.

Table 1. Closing price quotes of Treasury strips from *The Wall Street Journal* on November 10, 1989

Maturity	Bid	Ask
Aug 90	94.22	94.31
Nov 90	92.38	92.47
Nov 92	79.03	79.31
Nov 94	67.78	68.16
Nov 96	57.44	57.91
Nov 99	45.31	45.84
Nov 09	20.56	21.06
Nov 18	10.91	11.28

Table 2. The initial forward rate curve on November 10, 1989

Interval	Rate
[0,1)	7.773
[1,3)	7.738
[3,5)	7.629
[5,7)	8.210
[7,10)	7.846
[10,20)	7.839
[20,∞)	6.992

[4]The computed forward rates were obtained from the equation

$$f(0,T) = \ln(P(0,T)/P(0,T+\Delta))\frac{1}{\Delta}$$

where Δ represents the time interval between two consecutive zero-coupon bond price maturities.

To verify the accuracy of this forward rate curve, these rates were used to price the same stripped Treasuries; the results are given in Table 3. As expected, the correspondence is quite close.

To further verify these rates, we computed the prices of several Treasury bonds. These are reported in Table 4.

The model prices are not in exact agreement with market prices, although the fit is reasonable. A more careful specification of a forward curve would produce a better match. Since our intent is to illustrate the issues involved rather than to fit the model precisely, we shall ignore these differences for now. However, since the forward rate curve represents the initial state for our model, deviations in bond prices at this stage will lead to deviations in computed values later.

We now consider a Treasury bond futures contract. Most of the deliverable bonds for these contracts are callable. To simplify the analysis, as a first step in the computations, we modeled them as noncallable and as maturing at the first callable date. We investigate the March 1990 contract. The deliverables we used for the computations are given in Table 5.

The model prices we compute based on the forward rate curve in Table 2 are also contained in Table 5. These are consistently above the market prices. This probably reflects the fact that the callable bond should be worth less than the shorter noncallable bond we used to model it. However, to compute futures prices, we would like to match market prices exactly. To permit exact matching, our computer program allows the specification of a "yield spread" for each bond. Although this "correction" is not in character with the theoretical model, it does allow us to capture potential imperfections in the markets. To equate prices, we used the yield spreads (in percent per year) given in Table 6.

Next, we compute the futures price on the March 1990 contract. All bonds except the first on our list (as well as some others that we have not considered here) are deliverable. To compute the futures prices, we need to specify the volatility functions of the forward rate curve.

Table 3. Comparison of the actual Treasury stripped prices on November 10, 1989, with those computed from the forward rate curve in Table 2

Maturity	(Bid + Ask)/2	Model price
Aug 90	94.266	94.251
Nov 90	92.422	92.423
Nov 92	79.172	79.173
Nov 94	67.969	67.963
Nov 96	57.672	57.675
Nov 99	45.578	45.578
Nov 09	20.813	20.815
Nov 18	11.094	11.094

Table 4. Comparison of Treasury bond prices on November 10, 1989, with those computed based on the forward rate curve in Table 2

Coupon	Maturity	Model price	Market price Bid	Ask
8.250	May 90	100.19	100.19	100.38
7.250	Aug 92	98.41	98.41	98.59
8.625	Aug 93	102.38	102.06	102.38
10.50	Feb 95	111.09	110.66	110.84
11.75	Feb 01	127.63	127.25	127.44
13.75	Aug 04	149.47	148.84	149.03
8.750	May 17k*	109.53	108.72	108.84

*The symbol k means that for nonresident aliens, these bonds are exempt from withholding taxes.

Table 5. Closing Treasury bonds quotes from *The Wall Street Journal* on November 10, 1989

Issue		Model price	Market price Bid	Ask
11.750	Feb 05-10	132.88	131.94	132.13
10.000	May 05-10	117.75	117.00	117.19
12.750	Nov 05-10	142.66	141.56	141.75
13.875	May 06-11	153.56	152.44	152.63
14.000	Nov 06-11	155.47	154.34	154.53
10.375	Nov 07-12	122.63	121.75	121.94
12.000	Aug 08-13	138.69	137.69	137.88
13.250	May 09-14	151.66	150.66	150.84

Table 6. Yield adjustments needed to match Treasury bond prices on November 10, 1989, with model prices

Issue		Spread	Model price	Market price Bid	Ask
11.750	Feb 05-10	.0743	132.03	131.94	132.13
10.000	May 05-10	.0651	117.09	117.00	117.19
12.750	Nov 05-10	.0833	141.66	141.56	141.75
13.875	May 06-11	.0789	152.53	152.44	152.63
14.000	Nov 06-11	.0776	154.44	154.34	154.53
10.375	Nov 07-12	.0689	121.84	121.75	121.94
12.000	Aug 08-13	.0699	137.78	137.69	137.88
13.250	May 09-14	.0638	150.75	150.66	150.84

As a starting point, we used the historical volatility functions obtained by BARRA using data through May 1989. These volatility functions were obtained using a procedure similar to that outlined in Section 6. They used principal-components analysis to determine the first two principal components of the perturbations of the forward rate curve. For our program, these functions need to be specified at several points; the program then computes values by linear interpolation. The BARRA volatility functions are given in Table 7. The first factor describes a term structure shift that is not parallel. Short rates move more than long rates. The secondary factor describes a term structure twist.

These discretized volatility functions were used for the proportional volatility model, that is, they were multiplied by the forward rate (and the square root of the length of time passing, measured in years) to get the change in the forward rate over a period. (If the forward rate was greater than 1, i.e., 100 percent, the value 1 was used in place of the forward rate.) The value of the futures contract was computed to be 99.13. The actual market price was 99.38. Thus, our model underpriced the futures contract (or the market overpriced it).

Table 7. BARRA estimates of the proportionate volatility functions given in expression (17), based on data through May 1989

Time to maturity (τ)	$\sigma^1(\tau)$	$\sigma^2(\tau)$
0	.2393	−.0793
1	.2078	−.0429
3	.1767	−.0262
5	.1665	−.0049
7	.1494	.0164
10	.1331	.0443
20	.1278	.0804
30	.1079	.1435

In computing the futures price, only four of these bonds were actually ever delivered (across all possible states). These were the November 05-10, November 06-11, November 07-12, and May 09-14. In computing the option values on the Treasury bond futures, we therefore restricted our consideration to only these four deliverable bonds. The futures options prices computed by our program are given in Table 8.

Table 8. The March 1990 Treasury bond futures option's model and market prices on November 10, 1989

	Strike	Model	Market
Calls	96	4.26	3.95
	98	2.93	2.64
	100	1.89	1.56
	102	1.21	0.89
	104	0.72	0.48
	106	0.23	0.25
Puts	96	1.20	0.72
	98	1.83	1.31
	100	2.75	2.22
	102	4.02	3.50
	104	5.49	5.05
	106	7.02	6.81

Most of these options were overpriced by our model, indicating that the volatility was probably too high. Recall that the volatilities were estimated with data preceding May 1989. Since it is plausible that the volatility changed between the period used to determine the $(\sigma_1(\tau), \sigma_2(\tau))$ functions and November 10, we make a simple adjustment to the $(\sigma_1(\tau), \sigma_2(\tau))$ functions. (This adjustment is similar in spirit to the use of implicit volatilities.) We multiply both $(\sigma_1(\tau), \sigma_2(\tau))$ curves by the same constant. Trial and error yielded a good constant of 0.82. The adjusted price for the futures is 99.17, and the resulting option prices are in Table 9.

Table 9. The March 1990 Treasury bond futures option's model and market prices on November 10, 1989, with adjusted volatilities

	Strike	Model	Market
Calls	96	3.94	3.95
	98	2.55	2.64
	100	1.51	1.56
	102	0.87	0.89
	104	0.38	0.48
	106	0.02	0.25
Puts	96	0.83	0.72
	98	1.41	1.31
	100	2.32	2.22
	102	3.64	3.50
	104	5.11	5.05
	106	6.83	6.81

The model prices in Table 9 are similar to the market prices. We remark, however, that the remaining differences are partly due to the fact that the model futures price is different from the market futures price.[5]

A major advantage of our approach to pricing and hedging interest rate sensitive contingent claims is that a single forward rate curve model will simultaneously price and hedge different time horizon options as well as different types of financial instruments. To demonstrate this, we used the same model to price both another set of futures and futures options, and some callable Treasury bonds.

Using the model given above, the price for the June 1990 futures contract was 99.09. The market price on November 10 was 99.19. Table 10 presents the futures options prices.

As can be seen, the correspondence between market and model prices is reasonable.

Finally, we used the model to price the four callable Treasury bonds used as deliverables for the futures contract. To do this, we first computed the value of a noncallable bond with the same maturity as the callable bond. Second, we computed the value of a call on that bond (which can be exercised only

[5]Of course, one could also adjust these prices to incorporate any difference between the estimated and market futures prices. Let $C(g)$ be the Treasury bond futures option's value as a function of the futures price g. Let g^* be the "true" futures price. The adjustment is $C(g^*) = C(g) + (\partial C(g)/\partial g)[g^* - g]$.

during the last five years). Last, we subtracted the call's value from the noncallable bond to yield the callable bond's value. For example, to price the November 05-10 12.75 bond, we price a November 10 12.75 bond, we price a call on this bond, and we subtract the values. The results are contained in Table 11.

The differences between model and market prices in Table 11 are of the same order of magnitude as the earlier reported differences between model and market prices of noncallable bonds (see Table 4), reflecting the difficulty in fitting the initial forward rate curve.

Table 10. Comparison of the June 1990 Treasury bond futures option's model and market prices on November 10, 1989

	Strike	Model	Market
Calls	96	4.30	4.44
	98	3.17	3.22
	100	2.21	2.23
	102	1.34	1.53
	104	0.80	1.00
	106	0.46	0.67
Puts	96	1.32	1.41
	98	2.10	2.11
	100	3.06	3.06

Table 11. Comparison of callable Treasury bond's model and market prices on November 10, 1989

Coupon	Maturity	Non-callable	Call	Model	Bid	Ask
12.75	Nov 05-10	148.35	6.24	142.11	141.56	141.75
14.00	Nov 06-11	162.26	7.13	155.13	154.34	154.53
10.375	Nov 07-12	125.49	3.50	121.99	121.75	121.94
13.25	May 09-14	157.05	5.70	151.35	150.66	150.84

8. Summary

This paper describes the Heath, Jarrow, Morton approach for pricing contingent claims given a random evolution of interest rates. The procedure is illustrated for three different economies: a deterministic economy, an economy with one random factor, and an economy with two random factors. Preliminary, and nonexhaustive, estimates of the volatility parameters were provided. Furthermore, these were utilized to price the Chicago Board of Trade's Treasury bond futures and Treasury bond futures options. These preliminary results provide support for the model's validity, and suggest the need to provide a complete empirical investigation of the HJM methodology. This, however, awaits subsequent research.

Appendix: Derivation of the Correction Term in Expression (11)

From (10),

$$\frac{P(t, T)}{P(t, t + h)} = \tilde{E}_t(P(t + h, T)).$$

Substitution yields

$$e^{-\int_{t+h}^{T} f(t, u)du} = \tfrac{1}{2}\left[e^{-\int_{t+h}^{T} [f(t, u) + \delta(t, u)h]du}\right] \cdot \left[e^{-\int_{t+h}^{T} \sigma(t, u, f(t, u))\sqrt{h}\,du} + e^{+\int_{t+h}^{T} \sigma(t, u, f(t, u))\sqrt{h}\,du}\right]$$

$$e^{\int_{t+h}^{T} \delta(t, u)h du} = \cosh\left(\int_{t+h}^{T} \sigma(t, u, f(t, u))\sqrt{h}\,du\right).$$

Taking natural logarithms and differentiating gives the result. Q.E.D.

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